

Analytical Approximation of the Neutrino Oscillation Probabilities at large θ_{13}

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ABSTRACT: We present a simple analytical approximation to the neutrino oscillation probabilities in matter. The moderately large value of θ_{13} , recently discovered by the reactor experiments Daya Bay and RENO, limits the ranges of applicability of previous analytical approximations which relied on expanding in $\sin \theta_{13}$. In contrast, our approximation, which is applicable to all oscillations channels at all energies and baselines, works well for large θ_{13} . We demonstrate the accuracy of our approximation by comparing it to the exact numerical result, as well as the approximations of Cervera et al. and Asano and Minakata. We also discuss the utility of our approach in figuring out the required baseline lengths and neutrino energies for the oscillation probabilities to exhibit certain desirable features.

KEYWORDS: Neutrino, Oscillation Probability, Matter Effect

ARXIV EPRINT: [1302.6773](https://arxiv.org/abs/1302.6773)

Contents

1	Introduction	2
2	The Approximation	3
2.1	Diagonalization of the Effective Hamiltonian	3
2.2	Effective Mixing Angles	4
2.3	Definition of β	5
2.4	Neutrino Case	5
2.5	Anti-Neutrino Case	8
2.6	Comment on Coding	8
3	Demonstration of the Accuracy of the Approximation	8
4	Applications	12
4.1	Determination of the Mass Hierarchy from ν_e Oscillations	12
4.2	The “Magic” Baseline	14
5	Summary	16
A	Conventions, Notation, and Basic Formulae	17
A.1	The PMNS Matrix	17
A.2	Neutrino Oscillation	18
A.3	Matter Effects	20
B	Jacobi Method	21
B.1	Setup	21
B.2	Diagonalization of a 2×2 Matrix	22
B.3	Neutrino Case	22
B.3.1	Mixing Angles and Mass-squared Differences	22
B.3.2	First rotation	23
B.3.3	Second rotation	25
B.3.4	Absorption of ϕ' into θ_{13}	27
B.4	Anti-Neutrino Case	28
B.4.1	First Rotation	28
B.4.2	Second Rotation	30
B.4.3	Absorption of $\bar{\phi}'$ into θ_{13}	32
C	Commutation of R_{13} and R_{23} through R_{12}	33
C.1	Neutrino Case	33
C.2	Anti-neutrino Case	35

1 Introduction

When performing long-baseline neutrino oscillation experiments on the Earth with accelerator based beams, or when detecting atmospheric neutrinos coming from below, the neutrinos necessarily traverse the Earth’s interior [1–7]. This makes the understanding of matter effects [8–10] on the neutrino oscillation probabilities an indispensable part of analyzing such experiments.

Though the exact three-flavor neutrino oscillation probabilities in constant-density matter can be expressed analytically,¹ the exact formula is too complicated to yield physical insight [12–15]. Consequently, various analytical approximations have been devised to better understand the physics potential of various neutrino experiments [9, 16–21]. These approximations relied on expansions in the small parameters $\alpha = \delta m_{21}^2 / \delta m_{31}^2 \approx 0.03$ and/or $s_{13} = \sin \theta_{13}$ in one form or another, a systematic treatment of which can be found in Ref. [19]. Of the formulae available, the one derived by Cervera et al. in Ref. [22] has been used frequently due to its simplicity. Unfortunately, the accuracy of the Cervera et al. formula suffers when the value of θ_{13} is as large as was measured by Daya Bay [23] and RENO [24], consistent with both earlier and later determinations by T2K [25], MINOS [26, 27], and Double Chooz [28, 29]. This is due to the fact that the Cervera et al. formula includes terms of order $O(\alpha^2)$, $O(\alpha s_{13})$, and $O(s_{13}^2)$. Given that the current world average of $s_{13} = \sin \theta_{13}$ is about 0.15 [30], not all terms of the same order are included. Asano and Minakata [31] have derived the order $O(\alpha s_{13}^2)$ and $O(s_{13}^4)$ corrections to the Cervera et al. formula, but the simplicity of the original expressions is lost.

In previous papers [32, 33], we had advocated the utility of the Jacobi method [34] in deriving approximate formulae for neutrino oscillation in matter. The formula presented there maintained the expressions for the oscillation probabilities in vacuum, and only replaced the mixings angles and the mass-squared differences with their effective values in matter. The approximation worked very well except when θ_{13} was very small, a possibility that could not be ignored until the Daya Bay/RENO measurements. In this paper, we reintroduce the method with further refinements which improve the accuracy of the approximation, while maintaining its ease of use.

This paper is organized as follows. In section 2, we explain our approach to the matter effect problem, and list all the formulae necessary to calculate our approximation. In section 3, we demonstrate the accuracy of our approximation at various baseline lengths, different mass hierarchies, and different values of the CP violating phase δ . Comparisons with the approximations of Cervera et al. [22] and Asano-Minakata [31] are also made. In section 4, we show how simple calculations using our approximation can be used to derive the baselines and energies at which the oscillation probabilities exhibit desirable features. We conclude in Section 5. Detailed derivation of our approximation is given in appendices B and C.

¹The cubic characteristic equation for the eigenvalues of the effective Hamiltonian can be solved analytically using Cardano’s formula [11].

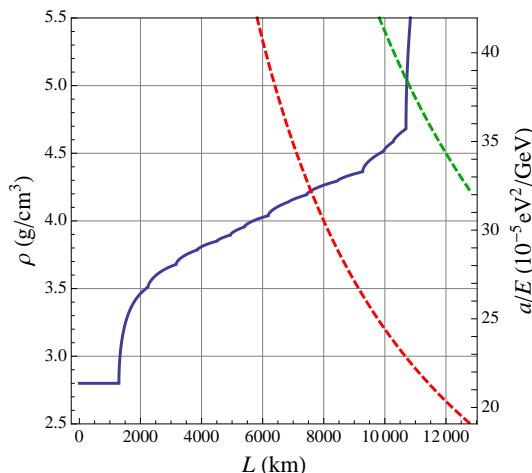


Figure 1. The dependence of the line-averaged mass density ρ on the baseline length L based on the Preliminary Reference Earth Model [35]. The labels on the right edge of the frame indicate the corresponding values of a/E . The green and red dashed lines indicate $\rho L = 54000 \text{ km} \cdot \text{g/cm}^3$ and $\rho L = 32000 \text{ km} \cdot \text{g/cm}^3$, respectively, which are conditions that will be discussed in section 4.

2 The Approximation

In the following, we use the conventions and notation reviewed in Appendix A.

2.1 Diagonalization of the Effective Hamiltonian

If the matter density along the baseline is constant, the effective Hamiltonian which governs the evolution of neutrino flavor in matter is given by

$$H_a = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.1)$$

where U is the neutrino mixing matrix in vacuum, and

$$a = 2\sqrt{2}G_F N_e E = 7.63 \times 10^{-5} (\text{eV}^2) \left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{E}{\text{GeV}} \right). \quad (2.2)$$

Here, N_e is the electron number density, ρ the matter mass density along the baseline, and E the neutrino energy. The above term appearing in the ee -component of H_a is due to the interaction of ν_e with the electrons in matter via W -exchange, and Eq. (2.2) assumes $N_e = N_p = N_n$ in Earth matter. It also assumes $E \ll M_W$ since the W -exchange interaction is approximated by a point-like four-fermion interaction. Z -exchange effects are flavor universal and only contribute a term proportional to the unit matrix to H_a , which can be dropped.

If we write the eigenvalues of H_a as λ_i ($i = 1, 2, 3$) and the diagonalization matrix as \tilde{U} , that is

$$H_a = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger, \quad (2.3)$$

then the neutrino oscillation probabilities in matter are obtained by simply taking their expressions in vacuum and replacing the elements of the mixing matrix U and the mass-square differences δm_{ij}^2 with their “effective” values in matter:

$$U_{\alpha i} \rightarrow \tilde{U}_{\alpha i}, \quad \delta m_{ij}^2 \rightarrow \delta \lambda_{ij} \equiv \lambda_i - \lambda_j. \quad (2.4)$$

Note that a is E -dependent, which means that both $\tilde{U}_{\alpha i}$ and $\delta \lambda_{ij}$ are also E -dependent. They also depend on the baseline length L since the average matter density ρ along a baseline varies with L . The L -dependence of the average ρ and the corresponding value of a/E are shown in Fig. 1.

For anti-neutrino beams, the flavor-evolution Hamiltonian in matter is

$$\bar{H}_a = U^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^T + \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.5)$$

In comparison to Eq. (2.1), the CP violating phase δ in U and the matter-effect term a both acquire minus signs. Let us write the eigenvalues of \bar{H}_a as $\bar{\lambda}_i$ ($i = 1, 2, 3$) and the diagonalization matrix as $\tilde{\bar{U}}$, that is

$$\bar{H}_a = \tilde{\bar{U}}^* \begin{bmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & \bar{\lambda}_2 & 0 \\ 0 & 0 & \bar{\lambda}_3 \end{bmatrix} \tilde{\bar{U}}^T. \quad (2.6)$$

Note that the tilde above $\tilde{\bar{U}}$ here is flipped to distinguish it from \tilde{U} in Eq. (2.3). The anti-neutrino oscillation probabilities in matter are then obtained by making the replacements

$$U_{\alpha i} \rightarrow \tilde{\bar{U}}_{\alpha i}, \quad \delta m_{ij}^2 \rightarrow \delta \bar{\lambda}_{ij} \equiv \bar{\lambda}_i - \bar{\lambda}_j, \quad (2.7)$$

in the vacuum expressions.

2.2 Effective Mixing Angles

As mentioned in the introduction, it is possible to write down exact analytical expressions for $\tilde{U}_{\alpha i}$ and $\delta \lambda_{ij}$, as well as their anti-neutrino counterparts [12]. However, simpler and more transparent approximate expressions can be obtained using the Jacobi method [34]. In our approach, the matrix \tilde{U} is approximated by taking the conventional parameterization of the matrix U in vacuum in terms of three mixing angles and a CP-violating phase, namely

$$U = R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0), \quad (2.8)$$

and replacing the values of θ_{12} and θ_{13} with their effective values in matter:²

$$\tilde{U} \approx R_{23}(\theta_{23}, 0) R_{13}(\theta'_{13}, \delta) R_{12}(\theta'_{12}, 0) . \quad (2.9)$$

The angle θ_{23} and the CP-violating phase δ are kept at their vacuum values. The eigenvalues λ_i ($i = 1, 2, 3$) of H_a are also given approximate expressions. Thus, to take matter effects into account when calculating neutrino oscillation probabilities, all that is necessary is to take their expressions in terms of the mixing angles and CP-phase in vacuum as is, and replace the two angles as well as the mass-squared differences with their effective values in matter: $\theta_{12} \rightarrow \theta'_{12}$, $\theta_{13} \rightarrow \theta'_{13}$, $\delta m_{ij}^2 \rightarrow \delta \lambda_{ij} = \lambda_i - \lambda_j$. This simplifies the calculation considerably, and allows for a transparent understanding of how matter-effects affect neutrino oscillation by looking at the a -dependence of the effective parameters.

Similarly, in the anti-neutrino case, we approximate

$$\tilde{U} \approx R_{23}(\theta_{23}, 0) R_{13}(\bar{\theta}'_{13}, \delta) R_{12}(\bar{\theta}'_{12}, 0) . \quad (2.10)$$

Again, θ_{23} and δ are unaffected while θ_{12} and θ_{13} are replaced by their effective values in matter. The eigenvalues $\bar{\lambda}_i$ ($i = 1, 2, 3$) of \bar{H}_a are given approximate expressions as in the neutrino case. Thus, the calculation of matter effects for anti-neutrino beams entails the replacements $\theta_{12} \rightarrow \bar{\theta}'_{12}$, $\theta_{13} \rightarrow \bar{\theta}'_{13}$, $\delta m_{ij}^2 \rightarrow \delta \bar{\lambda}_{ij} = \bar{\lambda}_i - \bar{\lambda}_j$.

2.3 Definition of β

In the following, we list the expressions necessary to calculate the effective mixing angles and the effective mass-squared differences for the neutrino and anti-neutrino cases separately. Detailed derivation of our approximation is given in Appendix B. We will also show plots depicting how various effective parameters depend on the matter-effect parameter a . Due to the wide separation in scale between δm_{21}^2 and δm_{31}^2 , we find it convenient to introduce the parameter β via³

$$\frac{a}{|\delta m_{31}^2|} = \varepsilon^{-\beta} , \quad \varepsilon \equiv \sqrt{\frac{\delta m_{21}^2}{|\delta m_{31}^2|}} \approx 0.17 , \quad (2.11)$$

and plot our effective parameters as functions of β instead of a . $\beta = 0$ corresponds to $a = |\delta m_{31}^2|$, $\beta = -2$ to $a = \delta m_{21}^2$, and so on.

2.4 Neutrino Case

For the neutrino case, the effective mixing angles in matter are given by

$$\begin{aligned} \tan 2\theta'_{12} &= \frac{(\delta m_{21}^2/c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2/c_{13}^2) \cos 2\theta_{12} - a} , \\ \tan 2\theta'_{13} &= \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - a} . \end{aligned} \quad (2.12)$$

²Absorbing matter effects into θ_{12} and θ_{13} is similar in spirit to Ref. [18].

³We avoid the use of the symbols α or A since they often respectively denote $\delta m_{21}^2/\delta m_{31}^2$ and $a/\delta m_{31}^2$ in the literature.

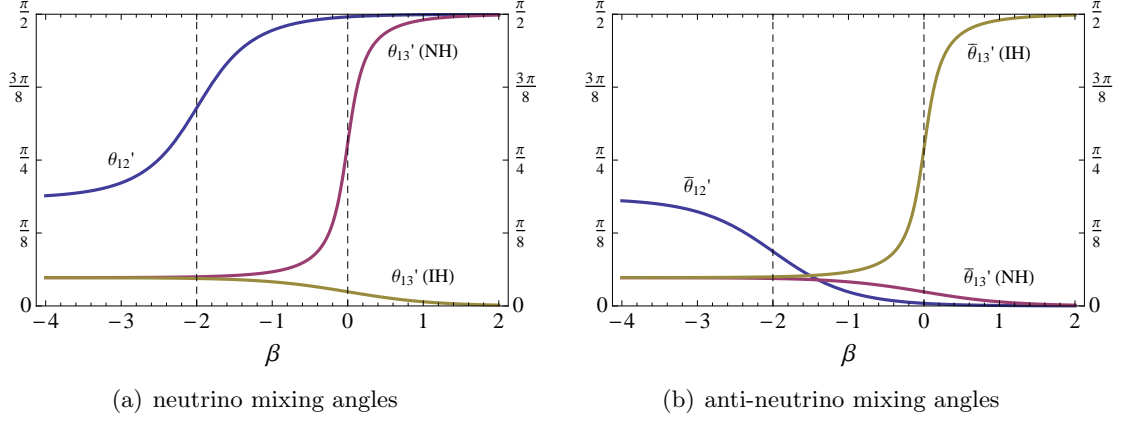


Figure 2. The dependences of the effective mixing angles on $\beta = -\log_\varepsilon(a/|\delta m_{31}^2|)$ for the neutrino (a) and antineutrino (b) cases. $\beta = 0$ corresponds to $a = |\delta m_{31}^2|$, and $\beta = -2$ to $a = \delta m_{21}^2$. The β -dependences of θ'_{13} and $\bar{\theta}'_{13}$ depend on the mass hierarchy: when $\delta m_{31}^2 > 0$ (normal hierarchy, NH) θ'_{13} increases toward $\pi/2$ whereas $\bar{\theta}'_{13}$ decreases toward zero, while in the $\delta m_{31}^2 < 0$ case (inverted hierarchy, IH), it is the other way around.

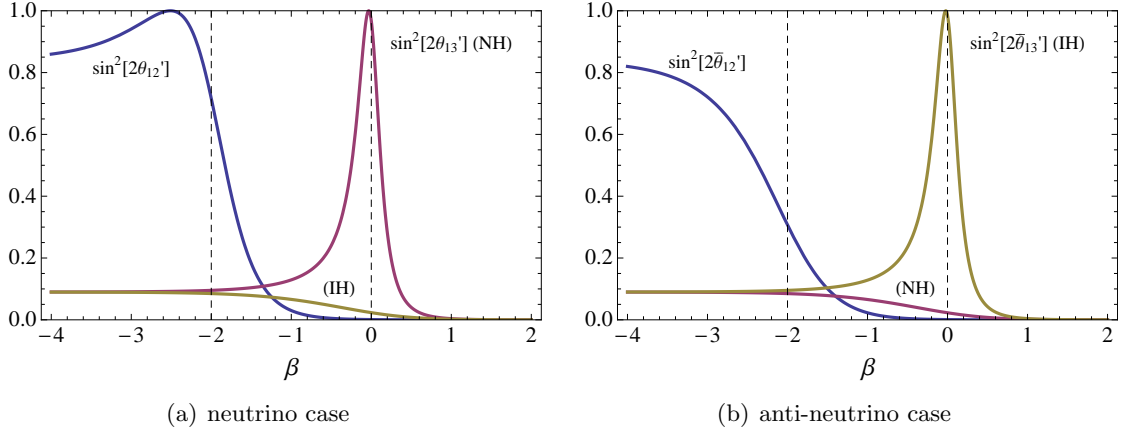


Figure 3. The β -dependences of the sines of twice the effective mixing angles for the neutrino (a) and antineutrino (b) cases. The difference in the behavior of the effective θ_{13} mixing angle for normal and inverted hierarchies will allow us to determine which is chosen by nature.

The β -dependence of these angles are shown in Fig. 2(a), and that of the sines of twice these angles in Fig. 3(a).

The eigenvalues of the effective Hamiltonian are approximated by

$$\begin{aligned}\lambda_1 &\approx \lambda'_- , \\ \lambda_2 &\approx \lambda''_{\mp} , \\ \lambda_3 &\approx \lambda''_{\pm} ,\end{aligned}\tag{2.13}$$

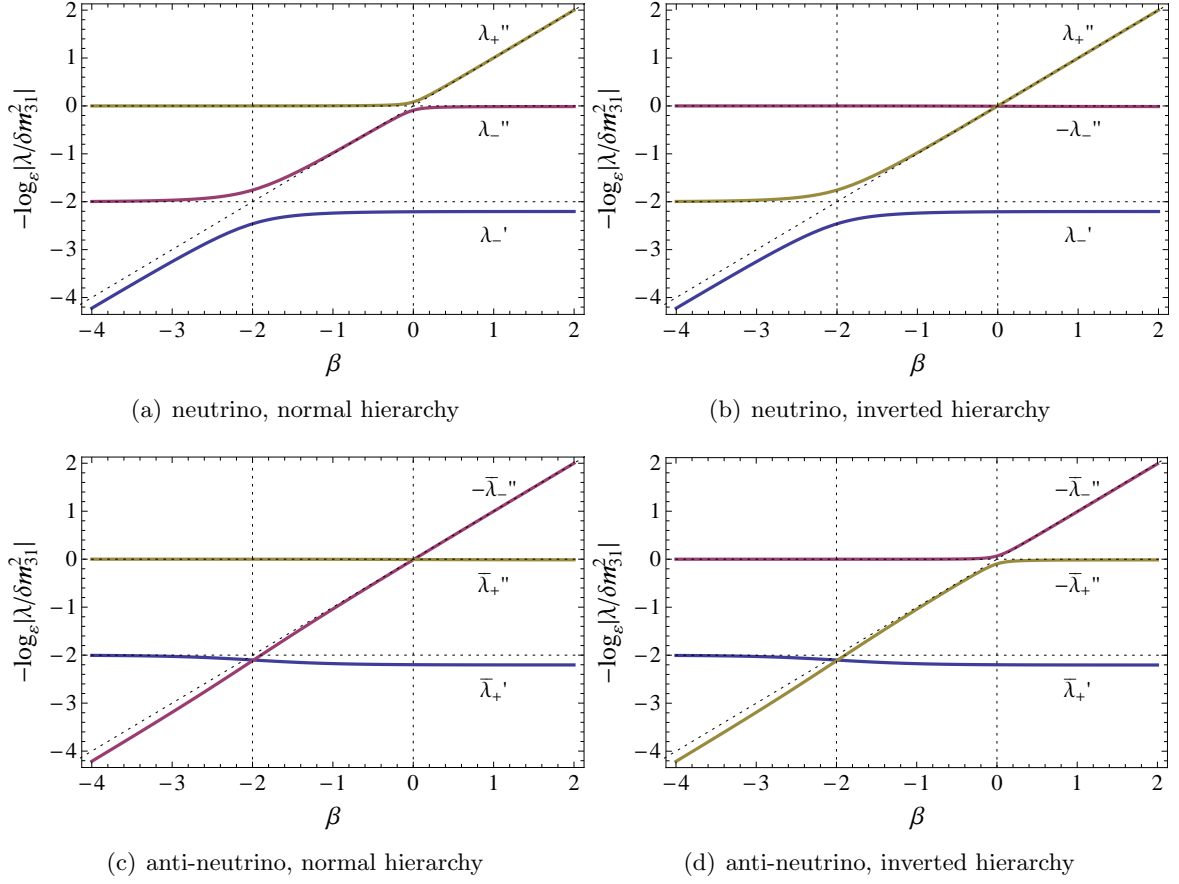


Figure 4. Dependence of the approximate eigenvalues of the effective Hamiltonian on $\beta = -\log_\varepsilon(a/|\delta m_{31}^2|)$ for the (a) neutrino normal hierarchy, (b) neutrino inverted hierarchy, (c) anti-neutrino normal hierarchy, and (d) anti-neutrino inverted hierarchy cases.

where the upper(lower) sign is for the normal(inverted) hierarchy, with

$$\begin{aligned}\lambda'_\pm &\equiv \frac{(\delta m_{21}^2 + ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 - ac_{13}^2)^2 + 4ac_{13}^2 s_{12}^2 \delta m_{21}^2}}{2}, \\ \lambda''_\pm &\equiv \frac{[\lambda'_+ + (\delta m_{31}^2 + as_{13}^2)] \pm \sqrt{[\lambda'_+ - (\delta m_{31}^2 + as_{13}^2)]^2 + 4a^2 s_{12}^2 c_{13}^2 s_{13}^2}}{2},\end{aligned}\quad (2.14)$$

and $s_{12}^{\prime 2} = \sin^2 \theta'_{12}$. The β -dependence of λ'_- and λ''_\pm for the normal and inverted mass hierarchies are shown in Figs. 4(a) and (b), respectively. That of λ'_+ is shown in Fig. 12(b). For the inverted hierarchy case, $\delta m_{31}^2 < 0$, the above expressions simplify to

$$\lambda_2 \approx \lambda''_+ \approx \lambda'_+, \quad \lambda_3 \approx \lambda''_- \approx \delta m_{31}^2 < 0. \quad (2.15)$$

2.5 Anti-Neutrino Case

For the anti-neutrino case, we need to flip the sign in front of the matter effect factor a , and the effective mixing angles are given by

$$\begin{aligned}\tan 2\bar{\theta}'_{12} &= \frac{(\delta m_{21}^2/c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2/c_{13}^2) \cos 2\theta_{12} + a} , \\ \tan 2\bar{\theta}'_{13} &= \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} + a} .\end{aligned}\quad (2.16)$$

The β -dependence of these angles are shown in Fig. 2(b), and that of the sines of twice these angles in Fig. 3(b).

The three eigenvalues of the effective Hamiltonian are approximated by

$$\begin{aligned}\bar{\lambda}_1 &\approx \bar{\lambda}_{\mp}'' , \\ \bar{\lambda}_2 &\approx \bar{\lambda}_{+}' , \\ \bar{\lambda}_3 &\approx \bar{\lambda}_{\pm}'' ,\end{aligned}\quad (2.17)$$

where the upper(lower) sign is for the normal(inverted) hierarchy, with

$$\begin{aligned}\bar{\lambda}'_{\pm} &\equiv \frac{(\delta m_{21}^2 - ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 + ac_{13}^2)^2 - 4ac_{13}^2 s_{12}^2 \delta m_{21}^2}}{2} , \\ \bar{\lambda}''_{\pm} &\equiv \frac{[\bar{\lambda}'_{-} + (\delta m_{31}^2 - as_{13}^2)] \pm \sqrt{[\bar{\lambda}'_{-} - (\delta m_{31}^2 - as_{13}^2)]^2 + 4a^2 \bar{c}_{12}^2 c_{13}^2 s_{13}^2}}{2} ,\end{aligned}\quad (2.18)$$

and $\bar{c}_{12}^2 = \cos^2 \bar{\theta}'_{12}$. The β -dependence of $\bar{\lambda}'_{+}$ and $\bar{\lambda}''_{\pm}$ for the normal and inverted mass hierarchies are shown in Figs. 4(c) and (d), respectively. That of $\bar{\lambda}'_{-}$ is shown in Fig. 15(b). For the normal hierarchy case, $\delta m_{31}^2 > 0$, the above expressions simplify to

$$\bar{\lambda}_1 \approx \bar{\lambda}_{-}'' \approx \bar{\lambda}_{-} , \quad \bar{\lambda}_3 \approx \bar{\lambda}_{+}'' \approx \delta m_{31}^2 . \quad (2.19)$$

2.6 Comment on Coding

As seen above, our formulae for both the neutrino and anti-neutrino cases are fairly compact and easy to code. In particular, the effective mixing angles for the neutrino and anti-neutrino cases can be calculated with the same code by simply flipping the sign of the matter-effect parameter a , *cf.* Eqs. (2.12) and (2.16). The same can be said of λ'_{\pm} and $\bar{\lambda}'_{\pm}$ defined in Eqs. (2.14) and (2.18). In the case of λ''_{\pm} and $\bar{\lambda}''_{\pm}$, one also needs to make the swap $\lambda'_{+} \leftrightarrow \bar{\lambda}_{-}$ but otherwise the code will be essentially the same. Once the effective mixing angles and the λ 's are obtained, one only needs to input them into the code for calculating the vacuum oscillation probabilities, and we are done.

3 Demonstration of the Accuracy of the Approximation

In this section, we demonstrate the accuracy of our approximation. For the vacuum values of the mixing angles and mass-squared differences, we use the global fit values from Ref. [30] listed in Table 1.

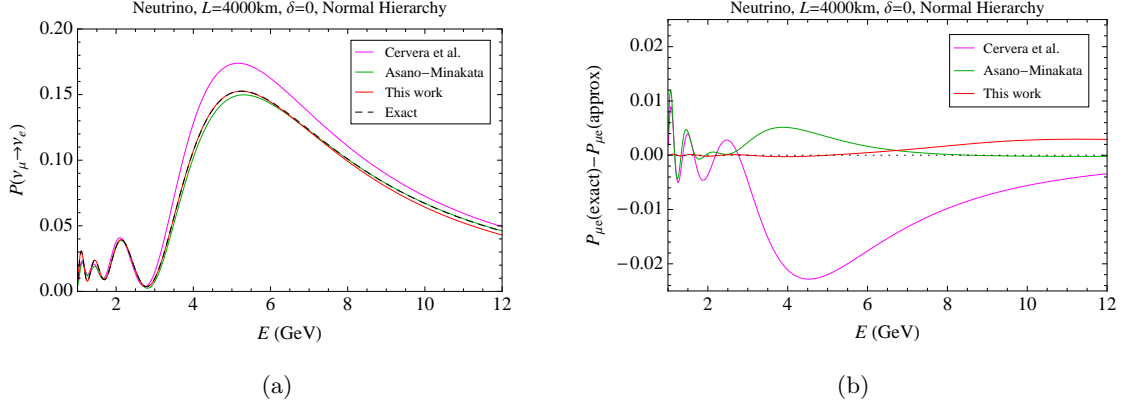


Figure 5. Comparison of the approximation formulae of Cervera et al., Asano-Minakata, and this work at $L = 4000\text{ km}$. The average matter density for this baseline length is $\rho = 3.81\text{ g/cm}^3$. The dashed line gives the exact numerical result. [35]

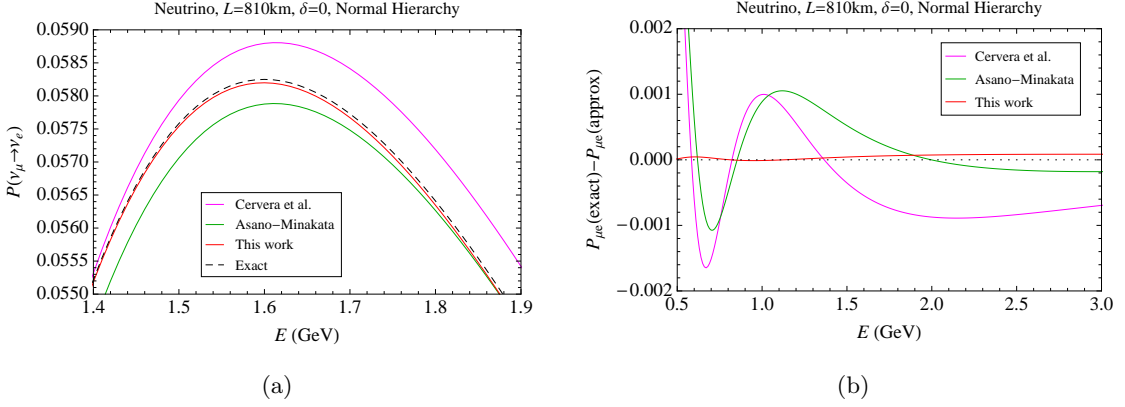


Figure 6. Comparison of the approximation formulae of Cervera et al., Asano-Minakata, and this work at $L = 810\text{ km}$, which is the distance from Fermilab to NO ν A. The average matter density for this baseline length is $\rho = 2.80\text{ g/cm}^3$. The dashed line gives the exact numerical result. [35]

We begin by comparing our approximation to those of Cervera et al. [22] and of Asano and Minakata [31]. In Fig. 5(a), we plot the approximate $\nu_\mu \rightarrow \nu_e$ oscillation probabilities calculated using the three approximations against the exact numerical result for the baseline length $L = 4000\text{ km}$. This is the distance used by Asano and Minakata in

δm_{21}^2	$7.5 \times 10^{-5}\text{ eV}^2$
δm_{31}^2	$2.47 \times 10^{-3}\text{ eV}^2$
$\sin^2 \theta_{23}$	0.5
$\sin^2 \theta_{12}$	0.3
$\sin^2 \theta_{13}$	0.023

Table 1. Best-fit values of oscillation parameters taken from Ref. [30].

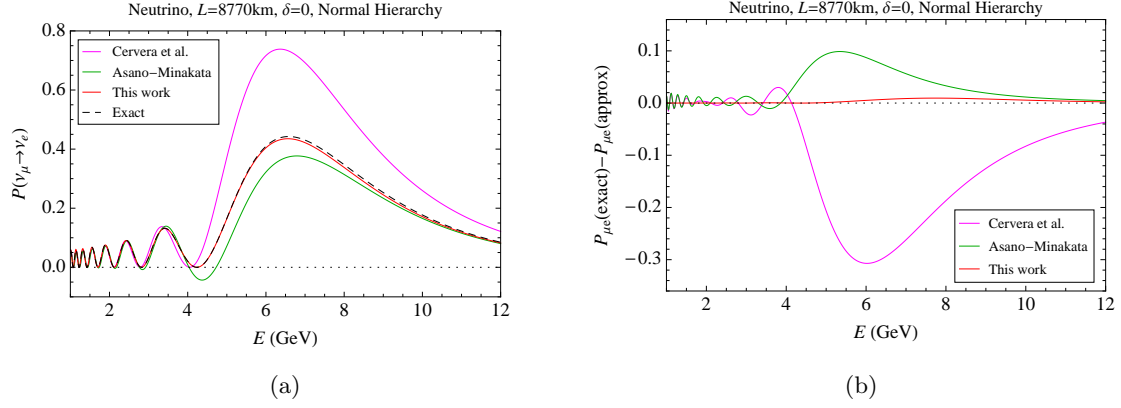


Figure 7. Comparison of the approximation formulae of Cervera et al., Asano-Minakata, and this work at $L = 8770$ km, which is the distance from CERN to Kamioka. The average matter density for this baseline length is $\rho = 4.33 \text{ g/cm}^3$. The dashed line gives the exact numerical result. Note that the Asano-Minakata formula gives negative probability for $E \sim 4$ GeV. [35]

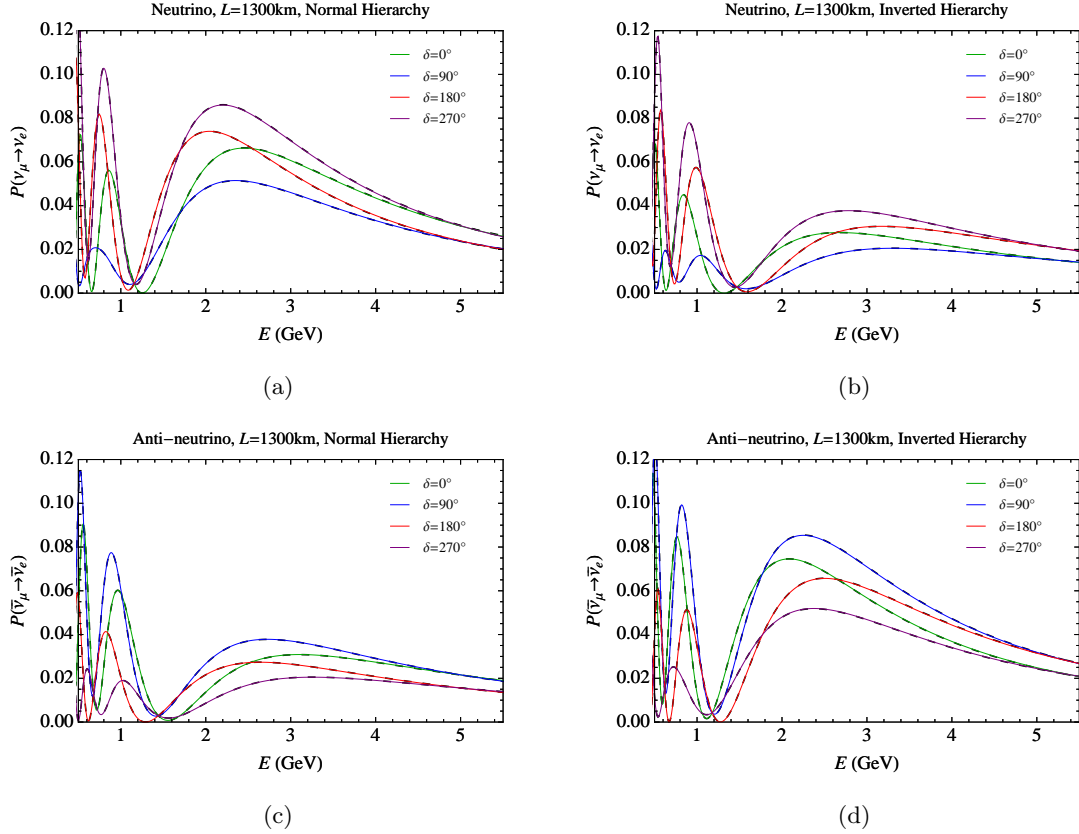


Figure 8. Comparison of our approximation formulae (colored) to the exact numerical results (black, dashed) for various values of the CP violating phase δ at $L = 1300$ km. The average matter density at this baseline length is $\rho = 2.87 \text{ g/cm}^3$.

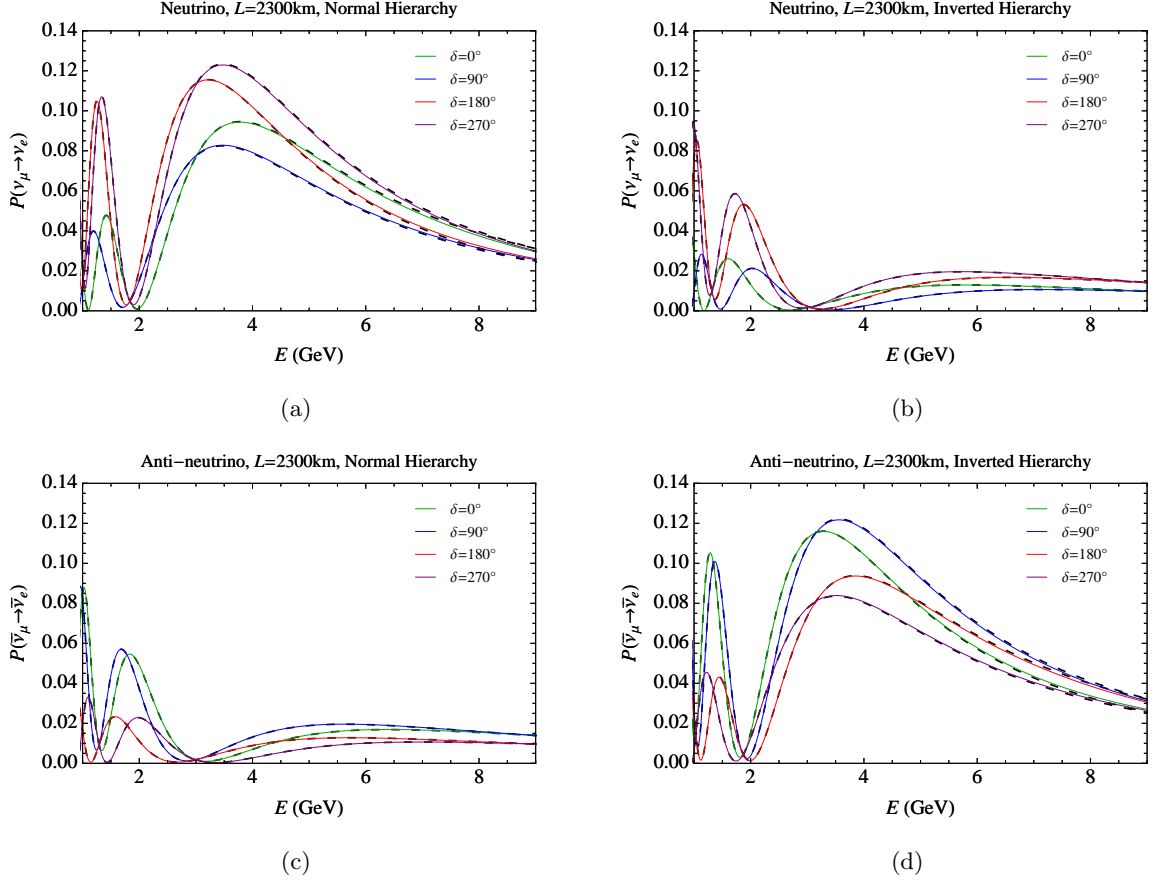


Figure 9. Comparison of our approximation formulae (colored) to the exact numerical results (black,dashed) for various values of the CP violating phase δ at $L = 2300$ km. The average matter density for $L = 2300$ km is $\rho = 3.54 \text{ g/cm}^3$.

Ref. [31] to demonstrate the strength of their formula. The average Earth matter density for this baseline is 3.81 g/cm^3 [35]. We consider the normal hierarchy case, $\delta m_{31}^2 > 0$, with the CP violating phase δ set to zero. The differences between the exact and approximate formulae are plotted in Fig. 5(b). As can be seen, at this baseline, both the Asano-Minakata formula and our approximation work much better than the Cervera et al. formula.

The comparison at a shorter baseline length of $L = 810$ km, which is the distance from Fermilab to NO ν A, is shown in Fig. 6. There, all three approximations work well, with our approximation being the most accurate.

The situation changes at the longer baseline length of $L = 8770$ km, which is the distance from CERN to Kamioka [36], as can be seen in Fig. 7. There, the Cervera et al. formula greatly overestimates $P(\nu_\mu \rightarrow \nu_e)$, while the Asano-Minakata formula leads to negative probability for $E \sim 4 \text{ GeV}$. In comparison, our approximation remains accurate.

The accuracy of our approximation for both the neutrino and anti-neutrino cases, and both mass hierarchies, for different values of the CP violating phase δ , is demonstrated in Figs. 8 and 9 for the two baselines $L = 1300$ km and $L = 2300$ km, respectively. These

distances correspond to those between Fermilab and Homestake (1300 km), and CERN and Pyh  salmi (2300 km) [37]. As is evident, our approximation maintains its accuracy for all energy ranges and mass densities.

4 Applications

4.1 Determination of the Mass Hierarchy from ν_e Oscillations

Consider the ν_e survival probability in matter which is given by

$$\begin{aligned}
P(\nu_e \rightarrow \nu_e) &= 1 - 4 |\tilde{U}_{e2}|^2 \left(1 - |\tilde{U}_{e2}|^2\right) \sin^2 \frac{\tilde{\Delta}_{21}}{2} - 4 |\tilde{U}_{e3}|^2 \left(1 - |\tilde{U}_{e3}|^2\right) \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + 2 |\tilde{U}_{e2}|^2 |\tilde{U}_{e3}|^2 \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31}\right) \\
&= 1 - 4 c_{13}'^2 s_{12}'^2 (1 - c_{13}'^2 s_{12}'^2) \sin^2 \frac{\tilde{\Delta}_{21}}{2} - \sin^2(2\theta_{13}') \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + s_{12}'^2 \sin^2(2\theta_{13}') \left(2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \frac{1}{2} \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31}\right) \\
&\xrightarrow{s_{12}' \approx 1} 1 - \sin^2(2\theta_{13}') \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} + \sin^2 \frac{\tilde{\Delta}_{31}}{2} - 2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} - \frac{1}{2} \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31}\right) \\
&= 1 - \sin^2(2\theta_{13}') \sin^2 \frac{\tilde{\Delta}_{32}}{2}, \tag{4.1}
\end{aligned}$$

where we have assumed that $a \gg \delta m_{21}^2$ so that $s_{12}' \approx 1$ is a good approximation. Similarly, we find:

$$\begin{aligned}
P(\nu_e \rightarrow \nu_\mu) &= 4 |U_{e2}|^2 |U_{\mu 2}|^2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4 |U_{e3}|^2 |U_{\mu 3}|^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + 2 \Re(U_{e3}^* U_{\mu 3} U_{e2} U_{\mu 2}^*) \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31}\right) \\
&\quad + 4 J_{(e,\mu)} \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin \tilde{\Delta}_{31} - \sin^2 \frac{\tilde{\Delta}_{31}}{2} \sin \tilde{\Delta}_{21}\right) \tag{4.2}
\end{aligned}$$

$$\begin{aligned}
&= 4 s_{12}'^2 c_{13}'^2 (c_{12}'^2 c_{23}'^2 + s_{12}'^2 s_{13}'^2 s_{23}'^2 - 2 s_{12}' c_{12}' s_{13}' c_{23}' s_{23}' \cos \delta) \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4 s_{13}'^2 c_{13}'^2 s_{23}'^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + 2 s_{12}' s_{13}' c_{13}'^2 s_{23}' (c_{12}' c_{23}' \cos \delta - s_{12}' s_{13}' s_{23}') \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31}\right) \\
&\quad - 4 s_{12}' c_{12}' s_{13}' c_{13}'^2 s_{23}' c_{23}' \sin \delta \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin \tilde{\Delta}_{31} - \sin^2 \frac{\tilde{\Delta}_{31}}{2} \sin \tilde{\Delta}_{21}\right) \tag{4.3}
\end{aligned}$$

$$\xrightarrow{s_{12}' \approx 1} s_{23}'^2 \sin^2(2\theta_{13}') \sin^2 \frac{\tilde{\Delta}_{32}}{2}, \tag{4.4}$$

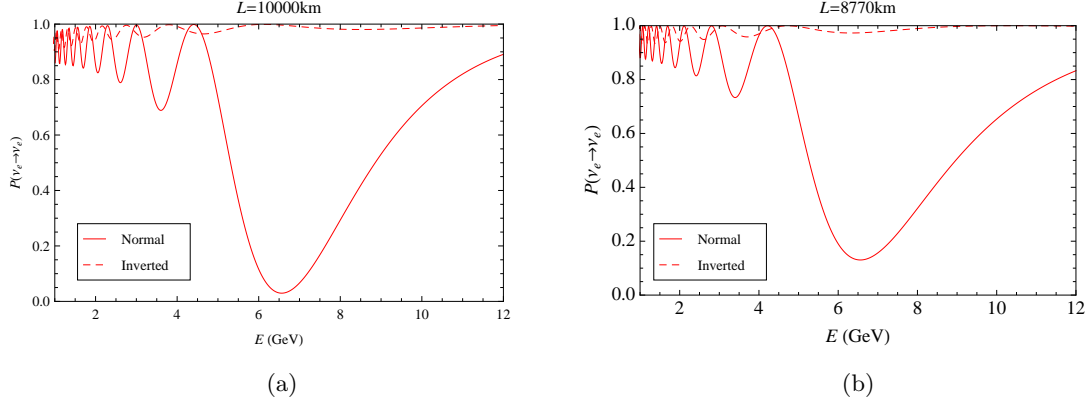


Figure 10. Comparison of the exact oscillation probabilities $P(\nu_e \rightarrow \nu_e)$ between the normal and inverted hierarchies at (a) $L = 10000$ km ($\rho = 4.53$ g/cm³), and (b) $L = 8770$ km ($\rho = 4.33$ g/cm³)

$$\begin{aligned}
P(\nu_e \rightarrow \nu_\tau) &= 4 |U_{e2}|^2 |U_{\tau 2}|^2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4 |U_{e3}|^2 |U_{\tau 3}|^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + 2 \Re(U_{e3}^* U_{\tau 3} U_{e2} U_{\tau 2}^*) \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) \\
&\quad + 4 J_{(e,\tau)} \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin \tilde{\Delta}_{31} - \sin^2 \frac{\tilde{\Delta}_{31}}{2} \sin \tilde{\Delta}_{21} \right) \\
&= 4 s_{12}'^2 c_{13}'^2 (c_{12}'^2 s_{23}'^2 + s_{12}'^2 s_{13}'^2 c_{23}'^2 - 2 s_{12}' c_{12}' s_{13}' s_{23}' c_{23}' \cos \delta) \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4 s_{13}'^2 c_{13}'^2 c_{23}'^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad - 2 s_{12}' s_{13}' c_{13}'^2 c_{23}' (c_{12}' s_{23}' \cos \delta + s_{12}' s_{13}' c_{23}') \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) \\
&\quad + 4 s_{12}' c_{12}' s_{13}' c_{13}'^2 s_{23}' c_{23}' \sin \delta \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin \tilde{\Delta}_{31} - \sin^2 \frac{\tilde{\Delta}_{31}}{2} \sin \tilde{\Delta}_{21} \right) \\
&\xrightarrow{s_{12}' \approx 1} c_{23}'^2 \sin^2(2\theta_{13}') \sin^2 \frac{\tilde{\Delta}_{32}}{2}. \tag{4.5}
\end{aligned}$$

From Fig. 3, it is clear that the factor $\sin^2(2\theta_{13}')$ in these expressions behaves quite differently depending on the mass hierarchy. For normal hierarchy $\sin^2(2\theta_{13}')$ will peak around $a \approx \delta m_{31}^2$ but for the inverted hierarchy case it will not. This will become manifest if the factor $\sin^2(\tilde{\Delta}_{32}/2)$ also peaked at or near the same energy,

For the normal hierarchy case, when $a \approx \delta m_{31}^2$ we have

$$\delta\lambda_{32} = \lambda_+'' - \lambda_-'' \approx \sqrt{[\lambda_+' - (\delta m_{31}^2 + a s_{13}^2)]^2 + 4a^2 c_{13}^2 s_{13}^2} \approx 2s_{13} a. \tag{4.6}$$

Therefore,

$$\frac{\tilde{\Delta}_{32}}{2} = \frac{\delta\lambda_{32}}{4E} L \approx \frac{s_{13} a}{2E} L = (2.9 \times 10^{-5}) \left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{L}{\text{km}} \right)$$

$$= \frac{\pi}{2} \left(\frac{\rho L}{54000 \text{ (km} \cdot \text{g/cm}^3)} \right). \quad (4.7)$$

From Fig. 1, it is clear that $\rho L < 54000 \text{ (km} \cdot \text{g/cm}^3)$ as long as the neutrino beam does not enter the core of the Earth, at which point the constant average matter density approximation breaks down. Therefore, in order to take $\tilde{\Delta}_{32}/2$ as close as possible to $\pi/2$ while preventing the beam from entering the Earth's core, we need $L \sim 10000 \text{ km}$.

For instance, if we take $L = 10000 \text{ km}$ for which $\rho = 4.53 \text{ g/cm}^3$, we have $\rho L \approx 45300 \text{ km} \cdot \text{g/cm}^3$. The value of $\tilde{\Delta}_{32}/2$ at resonance $a \approx \delta m_{31}^2$ is then

$$\frac{\pi}{2} \times \frac{45300}{54000} = 0.42 \pi, \quad (4.8)$$

leading to an oscillation peak/dip factor of $\sin^2(\tilde{\Delta}_{32}/2) = 0.94$. Using Eq. (2.2), the neutrino beam energy at which $a \approx \delta m_{31}^2$ is found to be

$$\frac{E}{\text{GeV}} = \frac{(\delta m_{31}^2/\text{eV}^2)}{(7.63 \times 10^{-5}) \times (\rho/(\text{g/cm}^3))} = \frac{(2.47 \times 10^{-3})}{(7.63 \times 10^{-5}) \times (4.53)} \approx 7. \quad (4.9)$$

Indeed, in Fig. 10(a) we show the exact ν_e survival probabilities at $L = 10000 \text{ km}$ for both hierarchies, and we can see that the normal hierarchy case dips by over 95% around $E \sim 6.5 \text{ GeV}$. Thus, our rough estimates give a highly reliable result.

If we take a somewhat shorter baseline of $L = 8770 \text{ km}$, which is the distance between CERN and Kamioka [36], we have $\rho = 4.33 \text{ g/cm}^3$, and $\rho L \approx 38000 \text{ km} \cdot \text{g/cm}^3$. The value of $\tilde{\Delta}_{32}/2$ at resonance $a \approx \delta m_{31}^2$ is then

$$\frac{\pi}{2} \times \frac{38000}{54000} = 0.35 \pi, \quad (4.10)$$

leading to an oscillation peak/dip factor of $\sin^2(\tilde{\Delta}_{32}/2) = 0.8$, which is still fairly prominent. Using Eq. (2.2), the neutrino beam energy at which $a \approx \delta m_{31}^2$ is found to be

$$\frac{E}{\text{GeV}} = \frac{(\delta m_{31}^2/\text{eV}^2)}{(7.63 \times 10^{-5}) \times (\rho/(\text{g/cm}^3))} = \frac{(2.47 \times 10^{-3})}{(7.63 \times 10^{-5}) \times (4.33)} \approx 7.5. \quad (4.11)$$

The actual oscillation peak occurs slightly off resonance around $E = 6.5 \text{ GeV}$ as can already be seen in Fig. 7. Comparison of $P(\nu_e \rightarrow \nu_e)$ at $L = 8770 \text{ km}$ with $\delta = 0$ for the normal and inverted hierarchies are shown in Fig. 10(b). $P(\nu_e \rightarrow \nu_\mu)$ is compared in Fig. 11(b).

The differences between the normal and inverted hierarchies for both baselines is manifest, indicating that measuring these oscillation probabilities at this baseline would allow us to determine the mass hierarchy easily. (We consider the dependence on the CP violating phase δ in the next section.) Eqs. (4.1), (4.4), and (4.5) also suggest that the measurement may provide a better determination of $\sin^2 \theta_{23}$.

4.2 The “Magic” Baseline

The “magic” baseline is the baseline at which the dependence of $P(\nu_e \rightarrow \nu_\mu)$ on the CP violating phase δ vanishes [38].⁴ Looking at Eq. (4.2), the only term without δ -dependence

⁴An illuminating discussion on the physical meaning of the “magic baseline” can be found in Ref. [39].

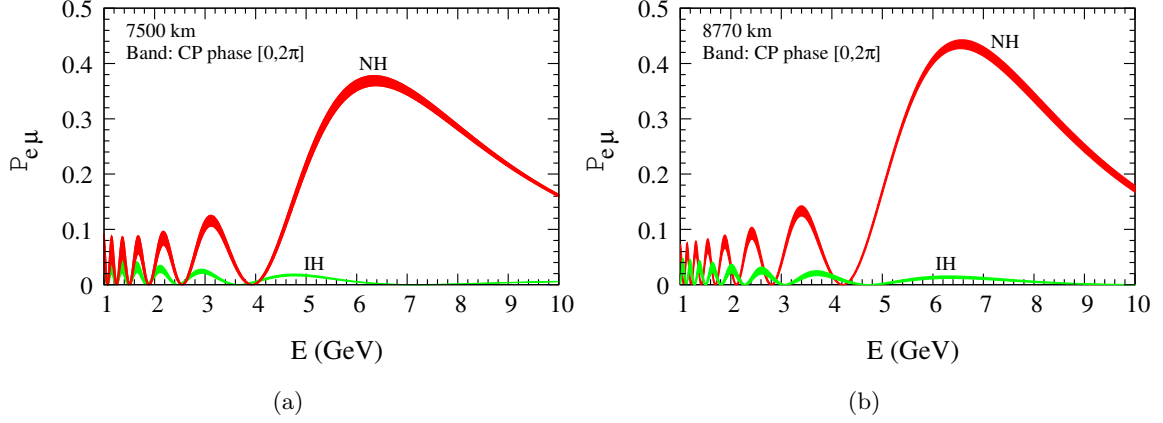


Figure 11. The dependence of the exact oscillation probability $P(\nu_e \rightarrow \nu_\mu)$ on the CP violating phase δ at (a) $L = 7500$ km ($\rho = 4.21$ g/cm³), and (b) $L = 8770$ km ($\rho = 4.33$ g/cm³) for the normal (red) and inverted (green) mass hierarchies.

is the $|\tilde{U}_{e3}|^2|\tilde{U}_{\mu 3}|^2$ term. To make every other term vanish, we must have

$$\sin \frac{\tilde{\Delta}_{21}}{2} = \sin \left(\frac{\delta \lambda_{21}}{4E} L \right) = 0. \quad (4.12)$$

Therefore, the magic baseline condition is

$$\frac{\delta \lambda_{21}}{4E} L = n\pi, \quad n \in \mathbb{Z}. \quad (4.13)$$

If we are in the energy and mass-density range such that $\delta m_{21}^2 < a < |\delta m_{31}^2|$, we can see from Fig. 4 that $\delta \lambda_{21} \approx a = 2\sqrt{2}G_F N_e E$, and the above condition reduces to

$$\sqrt{2}G_F N_e L \approx 2n\pi, \quad (4.14)$$

which is the usual magic baseline condition. Using Eq. (2.2), this condition for the $n = 1$ case becomes

$$\frac{\tilde{\Delta}_{21}}{2} \approx \frac{a}{4E} L = (9.7 \times 10^{-5}) \left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{L}{\text{km}} \right) = \pi, \quad (4.15)$$

that is

$$\frac{\rho L}{\text{km} \cdot \text{g/cm}^3} \approx 32000. \quad (4.16)$$

This is satisfied for $L \approx 7500$ km as can be read off of Fig. 1. Indeed, in Fig. 11(a) we plot the bands that $P(\nu_e \rightarrow \nu_\mu)$ at $L = 7500$ km sweeps for both mass hierarchies when δ is varied throughout its range of $[0, 2\pi]$. We can see that the dependence on δ is very weak.

However, if we look at Eq. (4.3) carefully, it is clear that all terms that include the CP violating phase δ are multiplied by c'_{12} which goes to zero when $a \gg \delta m_{21}^2$. Indeed, this was why δ did not appear in Eq. (4.4). The condition $a \gg \delta m_{21}^2$ demands

$$\left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{E}{\text{GeV}} \right) \gg 1, \quad (4.17)$$

which is clearly satisfied around the oscillation peak for the $L = 8770$ km case just discussed in the previous section. Thus, $P(\nu_e \rightarrow \nu_\mu)$ for this baseline is also only very weakly dependent on δ as shown in Fig. 11(b). We can conclude that, in general, as long as Eq. (4.17) is satisfied, one does not need to be at a specific “magic” baseline to suppress the δ -dependence of $P(\nu_e \rightarrow \nu_\mu)$.

5 Summary

We have presented a new and simple approximation for calculating the neutrino oscillation probabilities in matter. Our approximation was derived utilizing the Jacobi method [34], and we show in the appendix that at most two rotations are sufficient for approximate diagonalization of the effective Hamiltonian. The two rotation angles are absorbed into effective values of θ_{12} and θ_{13} .

As explained in detail in the appendix, the approximation works when $\theta_{13} = O(\varepsilon)$, where $\varepsilon = \sqrt{\delta m_{21}^2 / |\delta m_{31}^2|} = 0.17$, a condition which has been shown to be satisfied by Daya Bay [23] and RENO [24]. Our formulae are compact and can easily be coded as well as be manipulated by hand for back-of-the-envelope calculations. The application of our method to finding the $\nu_e \rightarrow \nu_\mu, \nu_\tau$ resonance conversion condition, and that to the determination of the ‘magic’ baseline [38, 39] have been demonstrated.

In this paper, only the matter effect due to Standard Model W exchange between the neutrinos and matter was considered. New Physics effects which distinguish between neutrino flavor would add extra terms to the effective Hamiltonian, which would require further rotations for diagonalization. This has been discussed previously in Ref. [33], and the potential constraints on New Physics from long baseline neutrino oscillations experiments in Refs. [40–42]. Updates to these works will be presented in future publications [43].

Acknowledgments

We would like to thank Minako Honda and Naotoshi Okamura for their contributions to the earlier version of this work [32, 33]. Helpful communications with T. Ohlsson, P. Roy, and K. Takeuchi are gratefully acknowledged. SKA was supported by the DST/INSPIRE Research Grant [IFA-PH-12], Department of Science & Technology, India. SKA is also grateful for the support of IFIC-CSIC, University of Valencia, Spain, where some initial portions of this work was carried out. TT was supported by the U.S. Department of Energy, grant number DE-FG05-92ER40677 task A, and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

A Conventions, Notation, and Basic Formulae

Here, we collect the basic formulae associated with neutrino oscillation in order to fix our notation and conventions.

A.1 The PMNS Matrix

Assuming three-generation neutrino mixing, the flavor eigenstates $|\nu_\alpha\rangle$ ($\alpha = e, \mu, \tau$) are related to the three mass eigenstates $|\nu_j\rangle$ ($j = 1, 2, 3$) via the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [44–46]

$$(V_{\text{PMNS}})_{\alpha j} \equiv \langle \nu_\alpha | \nu_j \rangle, \quad (\text{A.1})$$

that is,

$$\begin{aligned} |\nu_j\rangle &= \sum_{\alpha=e,\mu,\tau} |\nu_\alpha\rangle \langle \nu_\alpha | \nu_j \rangle = \sum_{\alpha=e,\mu,\tau} (V_{\text{PMNS}})_{\alpha j} |\nu_\alpha\rangle, \\ |\nu_\alpha\rangle &= \sum_{j=1,2,3} |\nu_j\rangle \langle \nu_j | \nu_\alpha \rangle = \sum_{j=1,2,3} (V_{\text{PMNS}})_{\alpha j}^* |\nu_j\rangle. \end{aligned} \quad (\text{A.2})$$

The standard parametrization is given by

$$V_{\text{PMNS}} = U \mathcal{P}, \quad (\text{A.3})$$

with⁵

$$\begin{aligned} U &= R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}, \\ \mathcal{P} &= \text{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2}). \end{aligned} \quad (\text{A.4})$$

Here, $R_{ij}(\theta, \delta)$ denotes a rotation matrix in the ij -plane of clockwise rotation angle θ with phases $\pm\delta$ on the off-diagonal ji and ij -elements, respectively, and $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$. Without loss of generality, we can adopt the convention $0 \leq \theta_{ij} \leq \pi/2$, $0 \leq \delta < 2\pi$ [47]. Of the six parameters in this expression and the three neutrino masses, which add up to a total of nine parameters, neutrino→neutrino oscillations are only sensitive to six:

- the three mixing angles: θ_{12} , θ_{23} , θ_{13} ,
- two mass-squared differences: δm_{21}^2 , δm_{31}^2 , where $\delta m_{ij}^2 = m_i^2 - m_j^2$, and
- the CP-violating phase: δ .

⁵Cervera et al. in Ref. [22] use a different convention in which the sign of δ is flipped.

The Majorana phases, α_{21} and α_{31} , only appear in lepton-number violating processes such as neutrinoless double beta decay, and cannot be determined via neutrino \rightarrow neutrino oscillations. The absolute scale of the neutrino masses also remain undetermined since neutrino oscillation is an interference effect.

A.2 Neutrino Oscillation

If a neutrino of flavor α is created at $x = 0$ with energy E , then the state of the neutrino at $x = 0$ is

$$|\nu_{\alpha,0}(x=0)\rangle = |\nu_{\alpha}\rangle = \sum_{j=1}^3 (V_{\text{PMNS}})_{\alpha j}^* |\nu_j\rangle . \quad (\text{A.5})$$

At $x = L$, the same state is

$$|\nu_{\alpha,0}(x=L)\rangle = \sum_{j=1}^3 e^{ip_j L} (V_{\text{PMNS}})_{\alpha j}^* |\nu_j\rangle = e^{ip_1 L} \sum_{j=1}^3 e^{i(p_j - p_1)L} (V_{\text{PMNS}})_{\alpha j}^* |\nu_j\rangle . \quad (\text{A.6})$$

Assing $m_j \ll E$ we can approximate

$$p_j = \sqrt{E^2 - m_j^2} = E - \frac{m_j^2}{2E} + \dots \quad (\text{A.7})$$

so that

$$p_j - p_1 \approx -\frac{\delta m_{j1}^2}{2E}, \quad \delta m_{j1}^2 = m_j^2 - m_1^2, \quad (\text{A.8})$$

and we find

$$|\nu_{\alpha,0}(x=L)\rangle = e^{ip_1 L} \sum_{j=1}^3 \exp\left(-i\frac{\delta m_{j1}^2}{2E}L\right) (V_{\text{PMNS}})_{\alpha j}^* |\nu_j\rangle . \quad (\text{A.9})$$

Therefore, the amplitude of observing the neutrino of flavor β at $x = L$ is given by (dropping the irrelevant overall phase)

$$\begin{aligned} \mathcal{A}_{\beta\alpha} &= \langle \nu_{\beta} | \nu_{\alpha,0}(x=L) \rangle \\ &= \left[\sum_{k=1}^3 \langle \nu_k | (V_{\text{PMNS}})_{\beta k} \right] \left[\sum_{j=1}^3 \exp\left(-i\frac{\delta m_{j1}^2}{2E}L\right) (V_{\text{PMNS}})_{\alpha j}^* |\nu_j\rangle \right] \\ &= \sum_{j=1}^3 (V_{\text{PMNS}})_{\beta j} \exp\left(-i\frac{\delta m_{j1}^2}{2E}L\right) (V_{\text{PMNS}})_{\alpha j}^* \\ &= \sum_{j=1}^3 U_{\beta j} \exp\left(-i\frac{\delta m_{j1}^2}{2E}L\right) U_{\alpha j}^* \\ &= \left[U \exp\left(-i\frac{\delta M^2}{2E}L\right) U^{\dagger} \right]_{\beta\alpha} \\ &= \left[\exp\left(-i\frac{H_0}{2E}L\right) \right]_{\beta\alpha}, \end{aligned} \quad (\text{A.10})$$

where

$$\delta M^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix}, \quad (\text{A.11})$$

and

$$H_0 = U \delta M^2 U^\dagger. \quad (\text{A.12})$$

Thus, the probability of oscillation from $|\nu_\alpha\rangle$ to $|\nu_\beta\rangle$ with neutrino energy E and baseline L is given by

$$\begin{aligned} P(\nu_\alpha \rightarrow \nu_\beta) &= |\mathcal{A}_{\beta\alpha}|^2 \\ &= \left| \sum_{j=1}^3 U_{\beta j} \exp\left(-i \frac{\delta m_{j1}^2}{2E} L\right) U_{\alpha j}^* \right|^2 \\ &= \delta_{\alpha\beta} - 4 \sum_{i>j} \Re(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin^2 \frac{\Delta_{ij}}{2} + 2 \sum_{i>j} \Im(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin \Delta_{ij}, \end{aligned} \quad (\text{A.13})$$

where⁶

$$\Delta_{ij} \equiv \frac{\delta m_{ij}^2}{2E} L = 2.534 \left(\frac{\delta m_{ij}^2}{\text{eV}^2} \right) \left(\frac{\text{GeV}}{E} \right) \left(\frac{L}{\text{km}} \right), \quad \delta m_{ij}^2 \equiv m_i^2 - m_j^2. \quad (\text{A.14})$$

Since

$$\Delta_{32} = \Delta_{31} - \Delta_{21}, \quad (\text{A.15})$$

only two of the three Δ_{ij} 's in Eq. (A.13) are independent. Eliminating Δ_{32} from Eq. (A.13) for the $\alpha = \beta$ case yields

$$\begin{aligned} P(\nu_\alpha \rightarrow \nu_\alpha) &= 1 - 4 |U_{\alpha 2}|^2 (1 - |U_{\alpha 2}|^2) \sin^2 \frac{\Delta_{21}}{2} - 4 |U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \sin^2 \frac{\Delta_{31}}{2} \\ &\quad + 2 |U_{\alpha 2}|^2 |U_{\alpha 3}|^2 \left(4 \sin^2 \frac{\Delta_{21}}{2} \sin^2 \frac{\Delta_{31}}{2} + \sin \Delta_{21} \sin \Delta_{31} \right), \end{aligned} \quad (\text{A.16})$$

and for the $\alpha \neq \beta$ case we have

$$\begin{aligned} P(\nu_\alpha \rightarrow \nu_\beta) &= 4 |U_{\alpha 2}|^2 |U_{\beta 2}|^2 \sin^2 \frac{\Delta_{21}}{2} + 4 |U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \frac{\Delta_{31}}{2} \\ &\quad + 2 \Re(U_{\alpha 3}^* U_{\beta 3} U_{\alpha 2} U_{\beta 2}^*) \left(4 \sin^2 \frac{\Delta_{21}}{2} \sin^2 \frac{\Delta_{31}}{2} + \sin \Delta_{21} \sin \Delta_{31} \right) \\ &\quad + 4 J_{(\alpha,\beta)} \left(\sin^2 \frac{\Delta_{21}}{2} \sin \Delta_{31} - \sin^2 \frac{\Delta_{31}}{2} \sin \Delta_{21} \right), \end{aligned} \quad (\text{A.17})$$

where $J_{(\alpha,\beta)}$ is the Jarlskog invariant [49]:

$$J_{(\alpha,\beta)} = +\Im(U_{\alpha 1}^* U_{\beta 1} U_{\alpha 2} U_{\beta 2}^*) = +\Im(U_{\alpha 2}^* U_{\beta 2} U_{\alpha 3} U_{\beta 3}^*) = +\Im(U_{\alpha 3}^* U_{\beta 3} U_{\alpha 1} U_{\beta 1}^*)$$

⁶Note that our notation differs from that of Cervera et al. in Ref. [22]. There, the symbol Δ_{ij} is defined without the factor of L , that is, $\Delta_{ij} = \delta m_{ij}^2/2E$. It also differs from that used by Freund in Ref. [17] where $\Delta = \delta m_{31}^2$, and $\hat{\Delta} = \delta m_{31}^2 L/4E$. Huber and Winter in Ref. [38] define $\Delta = \delta m_{31}^2 L/4E$, which is also used in Ref. [48]. So care is necessary when comparing formulae.

$$\begin{aligned}
&= -\Im(U_{\alpha 2}^* U_{\beta 2} U_{\alpha 1} U_{\beta 1}^*) = -\Im(U_{\alpha 1}^* U_{\beta 1} U_{\alpha 3} U_{\beta 3}^*) = -\Im(U_{\alpha 3}^* U_{\beta 3} U_{\alpha 2} U_{\beta 2}^*) \\
&= -J_{(\beta, \alpha)} .
\end{aligned} \tag{A.18}$$

In the parametrization given in Eq. (A.4), we have

$$J_{(\mu, e)} = -J_{(e, \mu)} = J_{(e, \tau)} = -J_{(\tau, e)} = J_{(\tau, \mu)} = -J_{(\mu, \tau)} = \hat{J} \sin \delta , \tag{A.19}$$

with

$$\hat{J} = s_{12} c_{12} s_{13} c_{13}^2 s_{23} c_{23} . \tag{A.20}$$

The oscillation probabilities for the anti-neutrinos are obtained by replacing $U_{\alpha i}$ with its complex conjugate, which only amounts to flipping the sign of δ in the parametrization of Eq. (A.4). It is clear from Eq. (A.16) that $P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\alpha) = P(\nu_\alpha \rightarrow \nu_\alpha)$, which is to be expected from the CPT theorem. For flavor changing oscillations, only the Jaroskog term in Eq. (A.17) changes sign.

A.3 Matter Effects

If the matter density along the baseline is constant, matter effects on neutrino oscillations can be taken into account by replacing the PMNS matrix elements and mass-squared differences with their “effective” values in matter:

$$\Delta_{ij} \rightarrow \tilde{\Delta}_{ij} , \quad U_{\alpha i} \rightarrow \tilde{U}_{\alpha i} , \tag{A.21}$$

where \tilde{U} is the unitary matrix that diagonalizes the modified Hamiltonian,

$$\begin{aligned}
H_a = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger = U \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix}}_{\substack{= \delta M^2 \\ = H_0}} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,
\end{aligned} \tag{A.22}$$

and

$$\tilde{\Delta}_{ij} = \frac{\delta \lambda_{ij}}{2E} L , \quad \delta \lambda_{ij} = \lambda_i - \lambda_j . \tag{A.23}$$

The factor a is due to the interaction of the $|\nu_e\rangle$ component of the neutrinos with the electrons in matter via W -exchange:

$$a = 2\sqrt{2} G_F N_e E . \tag{A.24}$$

Assuming $N_e = N_p \approx N_n$ in Earth matter, N_e for mass density per unit volume of ρ can be expressed using Avogadro’s number $N_A = 6.02214129 \times 10^{23} \text{ mol}^{-1}$ as

$$N_e = N_p \approx \rho N_A / 2 = (3.011 \times 10^{23} / \text{cm}^3) \times \left(\frac{\rho}{\text{g/cm}^3} \right) . \tag{A.25}$$

Thus, putting back powers of $\hbar c$ to convert from natural to conventional units, we find

$$a = 2\sqrt{2} G_F N_e E \times (\hbar c)^3$$

$$= (7.63 \times 10^{-5} \text{ eV}^2) \left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{E}{\text{GeV}} \right). \quad (\text{A.26})$$

For anti-neutrino beams, a is replaced by $-a$ in Eq. (A.22). Note that a is E -dependent, which means that both \tilde{U} and $\tilde{\Delta}_{ij}$ are also E -dependent. It is also assumed that $E \ll M_W$ since the W -exchange interaction is approximated by a point-like four-fermion interaction in deriving this expression.

B Jacobi Method

B.1 Setup

As mentioned in the introduction, it is possible to write down exact analytical expressions for $\tilde{\Delta}_{ij}$ and $\tilde{U}_{\alpha i}$ [12]. However, simpler and more transparent approximate expressions can be obtained using the Jacobi method as will be shown in the following.

We introduce the matrix

$$\mathcal{Q} = \text{diag}(1, 1, e^{i\delta}), \quad (\text{B.1})$$

and start with the partially diagonalized Hamiltonian:

$$\begin{aligned} H'_a &= \mathcal{Q}^\dagger U^\dagger H_a U \mathcal{Q} \\ &= \mathcal{Q}^\dagger \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + U^\dagger \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U \right\} \mathcal{Q} \\ &= \mathcal{Q}^\dagger \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} \mathcal{Q} + a \mathcal{Q}^\dagger \begin{bmatrix} U_{e1}^* U_{e1} & U_{e1}^* U_{e2} & U_{e1}^* U_{e3} \\ U_{e2}^* U_{e1} & U_{e2}^* U_{e2} & U_{e2}^* U_{e3} \\ U_{e3}^* U_{e1} & U_{e3}^* U_{e2} & U_{e3}^* U_{e3} \end{bmatrix} \mathcal{Q} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + a \begin{bmatrix} c_{12}^2 c_{13}^2 & c_{12} s_{12} c_{13}^2 & c_{12} c_{13} s_{13} \\ c_{12} s_{12} c_{13}^2 & s_{12}^2 c_{13}^2 & s_{12} c_{13} s_{13} \\ c_{12} c_{13} s_{13} & s_{12} c_{13} s_{13} & s_{13}^2 \end{bmatrix} \\ &= \begin{bmatrix} a c_{12}^2 c_{13}^2 & a c_{12} s_{12} c_{13}^2 & a c_{12} c_{13} s_{13} \\ a c_{12} s_{12} c_{13}^2 & a s_{12}^2 c_{13}^2 + \delta m_{21}^2 & a s_{12} c_{13} s_{13} \\ a c_{12} c_{13} s_{13} & a s_{12} c_{13} s_{13} & a s_{13}^2 + \delta m_{31}^2 \end{bmatrix}. \end{aligned} \quad (\text{B.2})$$

The matrix \mathcal{Q} serves to rid H'_a of any reference to the CP violating phase δ . The strategy we used in our previous papers [32, 33] was to approximately diagonalize H'_a through the Jacobi method using

$$\varepsilon = \sqrt{\frac{\delta m_{21}^2}{|\delta m_{31}^2|}} \approx 0.17, \quad (\text{B.3})$$

as the parameter to keep track of the sizes of the off-diagonal elements. We argued that approximate diagonalization was achieved when the off-diagonal elements were of order $O(\varepsilon^2 s_{13} |\delta m_{31}^2|)$.

Note that our ε differs from Asano and Minakata's ϵ in Ref. [31] where

$$\epsilon = \frac{\delta m_{21}^2}{|\delta m_{31}^2|} \approx 0.03. \quad (\text{B.4})$$

That is, $\epsilon = \varepsilon^2$. So care is necessary when comparing formulae.

B.2 Diagonalization of a 2×2 Matrix

Recall that for 2×2 real symmetric matrices, such as

$$M = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad (\text{B.5})$$

diagonalization is trivial. Just define

$$R = \begin{bmatrix} c_\omega & s_\omega \\ -s_\omega & c_\omega \end{bmatrix}, \quad \text{where} \quad c_\omega = \cos \omega, \quad s_\omega = \sin \omega, \quad \tan 2\omega \equiv \frac{2\beta}{\gamma - \alpha}, \quad (\text{B.6})$$

and we obtain

$$R^\dagger M R = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \quad (\text{B.7})$$

with

$$\begin{aligned} \Lambda_1 &= \frac{\alpha c_\omega^2 - \gamma s_\omega^2}{c_\omega^2 - s_\omega^2} = \frac{(\alpha + \gamma) \mp \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}, \\ \Lambda_2 &= \frac{\gamma c_\omega^2 - \alpha s_\omega^2}{c_\omega^2 - s_\omega^2} = \frac{(\alpha + \gamma) \pm \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}, \end{aligned} \quad (\text{B.8})$$

where the upper and lower signs are for the cases $\alpha < \gamma$ and $\alpha > \gamma$, respectively. The Jacobi method [34] entails iteratively diagonalizing 2×2 submatrices of a larger matrix in the order that requires the largest rotation angle at each step. In the limit of infinite iterations of this procedure, the matrix will converge to a diagonal matrix.

In the case of H'_a given in Eq. (B.2), at most two iterations are sufficient to achieve approximate diagonalization, neglecting off-diagonal elements of order $O(\varepsilon^2 s_{13} |\delta m_{31}^2|)$, regardless of the size of a . We demonstrate this in this appendix.

B.3 Neutrino Case

B.3.1 Mixing Angles and Mass-squared Differences

Let us first evaluate the sizes of the sines and cosines of the three vacuum mixing angles in comparison to the parameter ε defined in Eq. (B.3). The current best fit values for the mass-squared differences and mixing angles are listed in Table 1. The sines and cosines of the central values of the mixing angles are

$$\begin{aligned} s_{23} &= 0.71, & c_{23} &= 0.71, \\ s_{12} &= 0.55, & c_{12} &= 0.84, \\ s_{13} &= 0.15, & c_{13} &= 0.99. \end{aligned} \quad (\text{B.9})$$

Therefore, s_{13} is $O(\varepsilon)$ while all other sines and cosines are $O(1)$.

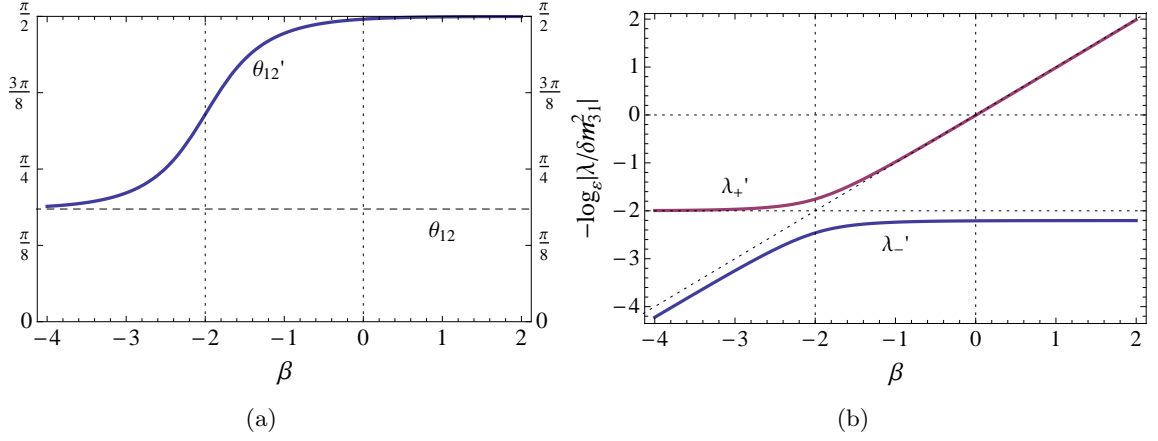


Figure 12. (a) The dependence of θ'_{12} on $\beta = -\log_\epsilon(a/|\delta m^2_{31}|)$. (b) The β -dependence of λ'_\pm .

B.3.2 First rotation

The effective hamiltonian we need to diagonalize is

$$\begin{aligned}
 H'_a &= \begin{bmatrix} ac_{12}^2 c_{13}^2 & ac_{12} s_{12} c_{13}^2 & ac_{12} c_{13} s_{13} \\ ac_{12} s_{12} c_{13}^2 & as_{12}^2 c_{13}^2 + \delta m_{21}^2 & as_{12} c_{13} s_{13} \\ ac_{12} c_{13} s_{13} & as_{12} c_{13} s_{13} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix} \\
 &= \begin{bmatrix} a\mathcal{O}(1) & a\mathcal{O}(1) & a\mathcal{O}(\epsilon) \\ a\mathcal{O}(1) & a\mathcal{O}(1) + \delta m_{21}^2 & a\mathcal{O}(\epsilon) \\ a\mathcal{O}(\epsilon) & a\mathcal{O}(\epsilon) & a\mathcal{O}(\epsilon^2) + \delta m_{31}^2 \end{bmatrix}. \quad (\text{B.10})
 \end{aligned}$$

Of the off-diagonal elements, the 1-2 element is the largest regardless of the size of a . Therefore, our first step is to diagonalize the 1-2 submatrix.

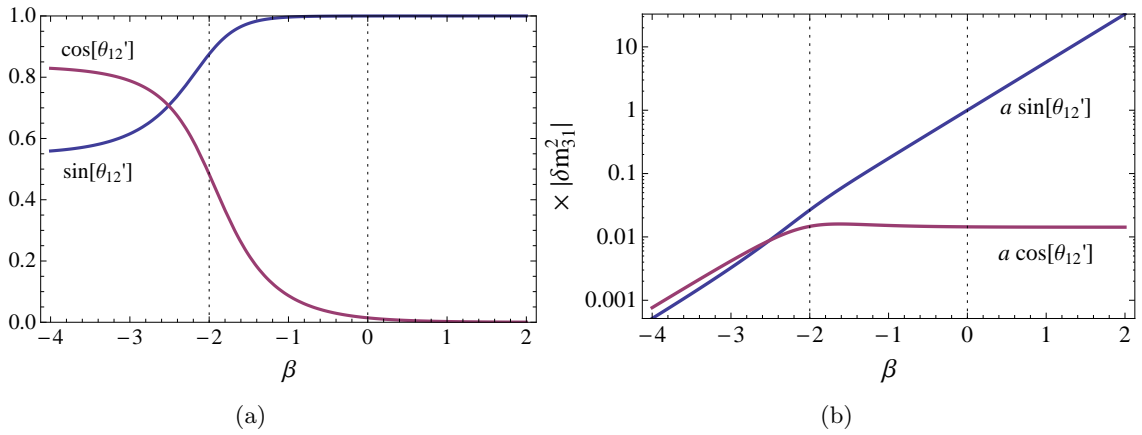


Figure 13. (a) The dependence of $s'_{12} = \sin \theta'_{12}$ and $c'_{12} = \cos \theta'_{12}$ on $\beta = -\log_\epsilon(a/|\delta m^2_{31}|)$. (b) The dependence of as'_{12} and ac'_{12} on β . The values are given in units of $|\delta m^2_{31}|$. The asymptotic value of ac'_{12} is $\delta m_{21}^2 s_{12} c_{12} / c_{13}^2 \approx 0.014 |\delta m^2_{31}| = \mathcal{O}(\epsilon^2 |\delta m^2_{31}|)$.

Define

$$V \equiv \begin{bmatrix} c_\varphi & s_\varphi & 0 \\ -s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{B.11})$$

where

$$c_\varphi = \cos \varphi, \quad s_\varphi = \sin \varphi, \quad \tan 2\varphi \equiv \frac{ac_{13}^2 \sin 2\theta_{12}}{\delta m_{21}^2 - ac_{13}^2 \cos 2\theta_{12}}, \quad \left(0 \leq \varphi \leq \frac{\pi}{2}\right). \quad (\text{B.12})$$

Using V , we find

$$H_a'' \equiv V^\dagger H_a' V = \begin{bmatrix} \lambda'_- & 0 & ac'_{12}c_{13}s_{13} \\ 0 & \lambda'_+ & as'_{12}c_{13}s_{13} \\ ac'_{12}c_{13}s_{13} & as'_{12}c_{13}s_{13} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix}, \quad (\text{B.13})$$

where

$$c'_{12} = \cos \theta'_{12}, \quad s'_{12} = \sin \theta'_{12}, \quad \theta'_{12} \equiv \theta_{12} + \varphi, \quad (\text{B.14})$$

and

$$\lambda'_\pm \equiv \frac{(\delta m_{21}^2 + ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 - ac_{13}^2)^2 + 4ac_{13}^2 s_{12}^2 \delta m_{21}^2}}{2}. \quad (\text{B.15})$$

The angle $\theta'_{12} = \theta_{12} + \varphi$ can be calculated directly without calculating φ via

$$\tan 2\theta'_{12} = \frac{\delta m_{21}^2 \sin 2\theta_{12}}{\delta m_{21}^2 \cos 2\theta_{12} - ac_{13}^2}, \quad \left(\theta_{12} \leq \theta'_{12} \leq \frac{\pi}{2}\right). \quad (\text{B.16})$$

The dependences of θ'_{12} and λ'_\pm on $\beta = -\log_\epsilon (a/|\delta m_{31}^2|)$ are plotted in Fig. 12. Note that θ'_{12} increases monotonically from θ_{12} to $\pi/2$ with increasing a . The β -dependence of $s'_{12} = \sin \theta'_{12}$ and $c'_{12} = \cos \theta'_{12}$ are shown in Fig. 13(a). For $a \gg \delta m_{21}^2$, s'_{12} and c'_{12} behave as

$$\begin{aligned} s'_{12} &= 1 - \frac{s_{12}^2 c_{12}^2}{2} \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^2 + \dots, \\ c'_{12} &= s_{12} c_{12} \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right) + s_{12} c_{12} (c_{12}^2 - s_{12}^2) \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^2 + \dots. \end{aligned} \quad (\text{B.17})$$

Therefore, for $a \gg \delta m_{21}^2$ we have $as'_{12} \approx a$ while $ac'_{12} \approx \delta m_{21}^2 s_{12} c_{12} / c_{13}^2 = \epsilon^2 |\delta m_{31}^2| s_{12} c_{12} / c_{13}^2 \approx 0.014 |\delta m_{31}^2| = O(\epsilon^2 |\delta m_{31}^2|)$. This behavior is shown in Fig. 13(b). Note that ac'_{12} never grows larger than $O(\epsilon^2 |\delta m_{31}^2|)$ for any a .

The values of λ'_\pm away from the level crossing point $a \sim \delta m_{21}^2$ for the $a \ll \delta m_{21}^2$ case are given by

$$\begin{aligned} \lambda'_- &= ac_{13}^2 c_{12}^2 \left[1 - s_{12}^2 \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right) - s_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right)^2 + \dots \right], \\ \lambda'_+ &= \delta m_{21}^2 \left[1 + s_{12}^2 \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right) + s_{12}^2 c_{12}^2 \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right)^2 + s_{12}^2 c_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right)^3 + \dots \right], \end{aligned} \quad (\text{B.18})$$

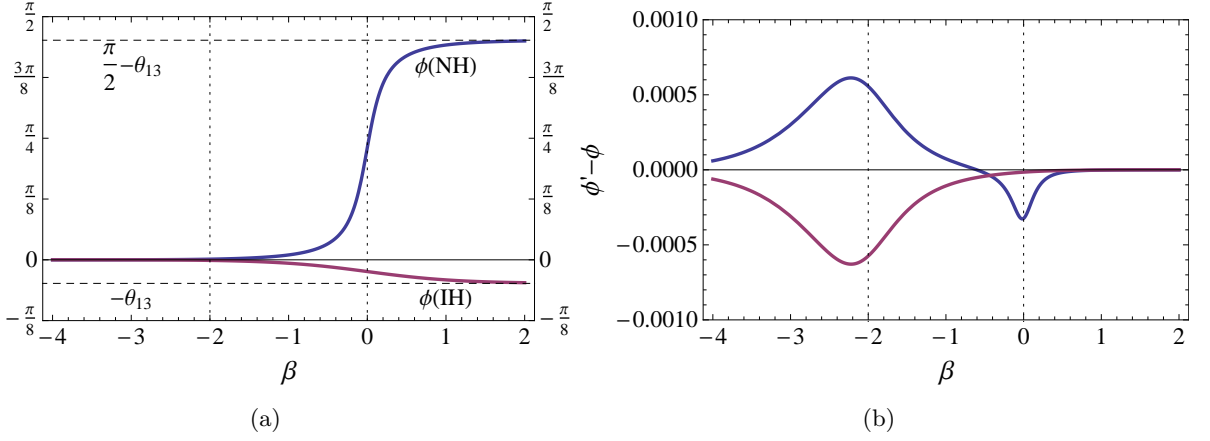


Figure 14. (a) The β -dependence of ϕ for the normal and inverted hierarchies. (b) The β -dependence of the difference $\phi' - \phi$.

and those for the $a \gg \delta m_{21}^2$ case by

$$\begin{aligned}\lambda'_- &= \delta m_{21}^2 c_{12}^2 \left[1 - s_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right) - s_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^2 + \dots \right], \\ \lambda'_+ &= ac_{13}^2 \left[1 + s_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right) + s_{12}^2 c_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^2 + s_{12}^2 c_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^3 + \dots \right].\end{aligned}\quad (\text{B.19})$$

We will use this expansion for λ'_+ later. Thus, the asymptotic values of λ'_\pm are $\lambda'_- \rightarrow ac_{13}^2 c_{12}^2$, $\lambda'_+ \rightarrow \delta m_{21}^2$ in the $a \rightarrow 0$ limit, and $\lambda'_- \rightarrow \delta m_{21}^2 c_{12}^2$, $\lambda'_+ \rightarrow ac_{13}^2$ in the $a \rightarrow \infty$ limit.

B.3.3 Second rotation

The effective hamiltonian after the first rotation was given by Eq. (B.13). When $a < \delta m_{21}^2$, both non-zero off-diagonal elements are of order $O(\varepsilon a) < O(\varepsilon^3 |\delta m_{31}^2|)$, since s'_{12} and c'_{12} are both $O(1)$ in that range as can be discerned from Fig. 13(a). However, as a increases beyond δm_{12}^2 and θ'_{12} approaches $\pi/2$, we have $as'_{12} \rightarrow a$, $ac'_{12} \rightarrow O(\varepsilon^2 |\delta m_{31}^2|)$ and the 2-3 element becomes the larger of the two. Therefore, a 2-3 rotation is needed next.

We define

$$W \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix}, \quad (\text{B.20})$$

where

$$c_\phi = \cos \phi, \quad s_\phi = \sin \phi, \quad \tan 2\phi \equiv \frac{as'_{12} \sin 2\theta_{13}}{\delta m_{31}^2 + as_{13}^2 - \lambda'_+}. \quad (\text{B.21})$$

The angle ϕ is in the first quadrant when $\delta m_{31}^2 > 0$, and in the fourth quadrant when $\delta m_{31}^2 < 0$. Then,

$$H_a''' \equiv W^\dagger H_a'' W = \begin{bmatrix} \lambda'_- & -ac'_{12}c_{13}s_{13}s_\phi & ac'_{12}c_{13}s_{13}c_\phi \\ -ac'_{12}c_{13}s_{13}s_\phi & \lambda''_{\mp} & 0 \\ ac'_{12}c_{13}s_{13}c_\phi & 0 & \lambda''_{\pm} \end{bmatrix}, \quad (\text{B.22})$$

where the upper(lower) sign corresponds to the normal(inverted) hierarchy case with

$$\lambda''_{\pm} \equiv \frac{[\lambda'_+ + (\delta m_{31}^2 + a s_{13}^2)] \pm \sqrt{[\lambda'_+ - (\delta m_{31}^2 + a s_{13}^2)]^2 + 4a^2 s_{12}'^2 c_{13}^2 s_{13}^2}}{2}. \quad (\text{B.23})$$

The β -dependences of λ''_{\pm} and ϕ are shown in Fig. 4 ((a) and (b)), and Fig. 14(a), respectively, for both mass hierarchies. For the normal hierarchy case, $\delta m_{31}^2 > 0$, the values of λ''_{\pm} away from the level crossing point $a \sim \delta m_{31}^2$ are approximately

$$\begin{aligned} \lambda''_+ &\approx \delta m_{31}^2 + a s_{13}^2, \\ \lambda''_- &\approx \lambda'_+, \end{aligned} \quad (\text{B.24})$$

when $a \ll \delta m_{31}^2$, and

$$\begin{aligned} \lambda''_+ &\approx a + s_{13}^2 \delta m_{31}^2 + c_{13}^2 s_{12}^2 \delta m_{21}^2, \\ \lambda''_- &\approx c_{13}^2 \delta m_{31}^2 + s_{13}^2 s_{12}^2 \delta m_{21}^2, \end{aligned} \quad (\text{B.25})$$

when $a \gg \delta m_{31}^2$. For the inverted hierarchy case, $\delta m_{31}^2 < 0$, where there is no level crossing, the values of λ''_{\pm} are approximately

$$\lambda''_- \approx \delta m_{31}^2 < 0, \quad \lambda''_+ \approx \lambda'_+, \quad (\text{B.26})$$

for all a .

At this point, we argue that the angle ϕ defined in Eq. (B.21) is well approximated by the angle ϕ' which we define via

$$\tan 2\phi' \equiv \frac{a \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) - a \cos 2\theta_{13}}. \quad (\text{B.27})$$

This approximation is obtained by first noting that ϕ is significantly different from zero only when $a \gg \delta m_{21}^2$. The expansion of λ'_+ in the denominator of the right-hand-side of Eq. (B.21) in powers of $\delta m_{21}^2/a$ was given in Eq. (B.19). Keeping only the first two terms, and noting also that $s'_{12} \approx 1$ to the same order when $a \gg \delta m_{21}^2$ (c.f. Eq. (??)) we obtain Eq. (B.27). The β -dependence of the difference $\phi' - \phi$ is plotted in Fig. 14(b), and we can see that the disagreement is at most $O(\varepsilon^4)$. Thus, we replace ϕ with ϕ' in the following.

Now, the effective Hamiltonian after the second rotation was given by Eq. (B.22). Note that all of the non-zero off-diagonal elements include the factor $a c'_{12}$, which is never larger than $O(\varepsilon^2 |\delta m_{31}^2|)$ regardless of the value of a as discussed above. They also all include a factor of s_{13} , which is $O(\varepsilon)$ as we have seen in Eq. (B.9). Therefore, all off-diagonal elements of H''_a are of order $O(\varepsilon^2 s_{13} |\delta m_{31}^2|) = O(\varepsilon^3 |\delta m_{31}^2|)$ or smaller regardless of the size of a . Note that had the value of s_{13} been smaller, the sizes of the neglected terms would have been proportionately smaller also. We conclude that, at this point, off-diagonal elements are negligible and a third rotation is not necessary.

B.3.4 Absorption of ϕ' into θ_{13}

From the above consideration, we conclude that the matrix which diagonalizes H'_a , Eq. (B.2), is given approximately by VW , and that the effective neutrino mixing matrix becomes

$$\tilde{U} \approx U Q V W = \underbrace{R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0)}_U \underbrace{Q}_{V} \underbrace{R_{12}(\varphi, 0) R_{23}(\phi', 0)}_W. \quad (\text{B.28})$$

Using

$$\begin{aligned} R_{12}(\theta_{12}, 0) Q &= Q R_{12}(\theta_{12}, 0), \\ R_{13}(\theta_{13}, \delta) Q &= Q R_{13}(\theta_{13}, 0), \end{aligned} \quad (\text{B.29})$$

we find

$$\begin{aligned} \tilde{U} &\approx R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0) Q R_{12}(\varphi, 0) R_{23}(\phi', 0) \\ &= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta_{12}, 0) R_{12}(\varphi, 0) R_{23}(\phi', 0) \\ &= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta_{12} + \varphi, 0) R_{23}(\phi', 0) \\ &= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta'_{12}, 0) R_{23}(\phi', 0). \end{aligned} \quad (\text{B.30})$$

Here, we argue that

$$R_{12}(\theta'_{12}, 0) R_{23}(\phi', 0) \approx R_{13}(\phi', 0) R_{12}(\theta'_{12}, 0), \quad (\text{B.31})$$

that is, the 2-3 rotation becomes a 1-3 rotation when commuted through $R_{12}(\theta'_{12}, 0)$. This is due to the fact that ϕ' only becomes non-negligible when $a \gg \delta m_{12}^2$ where $s'_{12} \approx 1$ and $c'_{12} \approx 0$, which means

$$R_{12}(\theta'_{12}, 0) \approx \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{B.32})$$

and it is straightforward to see that

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c'_\phi & s'_\phi \\ 0 & -s'_\phi & c'_\phi \end{bmatrix} = \begin{bmatrix} c'_\phi & 0 & s'_\phi \\ 0 & 1 & 0 \\ -s'_\phi & 0 & c'_\phi \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{B.33})$$

where $s'_\phi = \sin \phi'$ and $c'_\phi = \cos \phi'$. In the range $a \lesssim \delta m_{21}^2$, the angle ϕ' is very small and both $R_{23}(\phi', 0)$ and $R_{13}(\phi', 0)$ are approximately unit matrices and Eq. (B.31) is trivially satisfied. Curiously, this approximation breaks down around $a \sim \delta m_{31}^2$ for the normal hierarchy case when θ_{13} is $O(\varepsilon^2)$ or smaller, as is discussed in appendix C. However, given that the current experimentally preferred value of θ_{13} is $O(\varepsilon)$, the approximation is valid. Thus,

$$\begin{aligned} \tilde{U} &\approx R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{12}(\theta'_{12}, 0) R_{23}(\phi', 0) \\ &\approx R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13}, 0) R_{13}(\phi', 0) R_{12}(\theta'_{12}, 0) \\ &= R_{23}(\theta_{23}, 0) Q R_{13}(\theta_{13} + \phi', 0) R_{12}(\theta'_{12}, 0) \\ &= R_{23}(\theta_{23}, 0) Q R_{13}(\theta'_{13}, 0) R_{12}(\theta'_{12}, 0) \end{aligned}$$

$$= R_{23}(\theta_{23}, 0) R_{13}(\theta'_{13}, \delta) R_{12}(\theta'_{12}, 0) \mathcal{Q} , \quad (\text{B.34})$$

where we have defined

$$\theta'_{13} \equiv \theta_{13} + \phi' . \quad (\text{B.35})$$

This angle can be calculated directly without calculating ϕ' via

$$\tan 2\theta'_{13} = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - a} . \quad (\text{B.36})$$

The diagonal phase matrix \mathcal{Q} appearing rightmost in the above matrix product can be absorbed into the redefinition of the major phases and can be dropped. Thus, we arrive at our final approximation in which the vacuum mixing angles are replaced by their effective values in matter

$$\begin{aligned} \theta_{12} &\rightarrow \theta'_{12} = \theta_{12} + \varphi , \\ \theta_{13} &\rightarrow \theta'_{13} = \theta_{13} + \phi' , \\ \theta_{23} &\rightarrow \theta_{23} , \\ \delta &\rightarrow \delta , \end{aligned} \quad (\text{B.37})$$

and the eigenvalues of the effective Hamiltonian are given by

$$\begin{aligned} \lambda_1 &\approx \lambda'_- , \\ \lambda_2 &\approx \lambda''_{\mp} , \\ \lambda_3 &\approx \lambda''_{\pm} . \end{aligned} \quad (\text{B.38})$$

Note that of the mixing angles, only θ_{12} and θ_{13} are shifted. θ_{23} and δ stay at their vacuum values.

B.4 Anti-Neutrino Case

B.4.1 First Rotation

For the anti-neutrino case, the matter effect parameter a acquires a minus sign. Thus, the effective hamiltonian to be diagonalized is

$$\begin{aligned} \overline{H}'_a &= \begin{bmatrix} -ac_{12}^2 c_{13}^2 & -ac_{12} s_{12} c_{13}^2 & -ac_{12} c_{13} s_{13} \\ -ac_{12} s_{12} c_{13}^2 & -as_{12}^2 c_{13}^2 + \delta m_{21}^2 & -as_{12} c_{13} s_{13} \\ -ac_{12} c_{13} s_{13} & -as_{12} c_{13} s_{13} & -as_{13}^2 + \delta m_{31}^2 \end{bmatrix} \\ &= \begin{bmatrix} -a\mathcal{O}(1) & -a\mathcal{O}(1) & -a\mathcal{O}(\varepsilon) \\ -a\mathcal{O}(1) & -a\mathcal{O}(1) + \delta m_{21}^2 & -a\mathcal{O}(\varepsilon) \\ -a\mathcal{O}(\varepsilon) & -a\mathcal{O}(\varepsilon) & -a\mathcal{O}(\varepsilon^2) + \delta m_{31}^2 \end{bmatrix} . \end{aligned} \quad (\text{B.39})$$

The largest off-diagonal element is the 1-2 element. Therefore, our first step is to diagonalize the 1-2 submatrix.

Define

$$\overline{V} \equiv \begin{bmatrix} \bar{c}_\varphi & \bar{s}_\varphi & 0 \\ -\bar{s}_\varphi & \bar{c}_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad (\text{B.40})$$

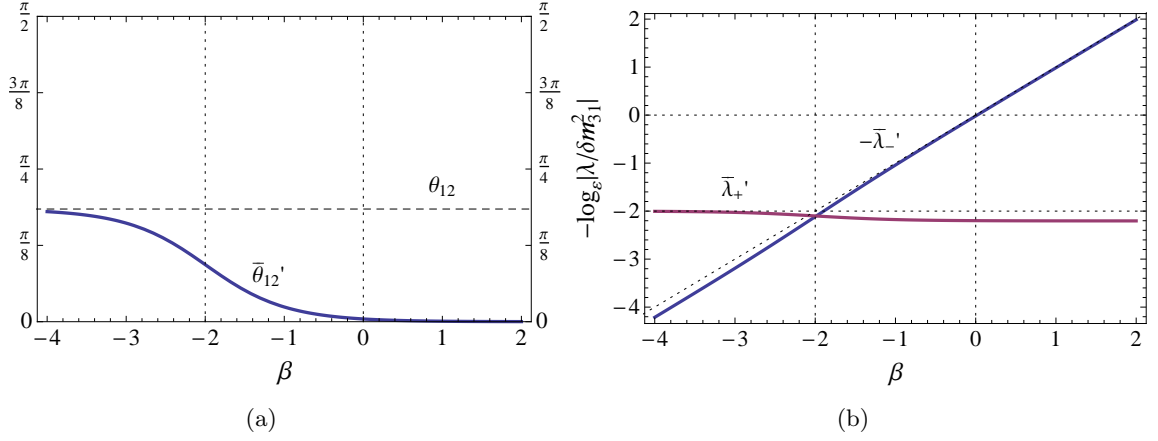


Figure 15. (a) The dependence of $\bar{\theta}'_{12}$ on $\beta = -\log_\varepsilon(a/|\delta m_{31}^2|)$. (b) The β -dependence of $\bar{\lambda}'_{\pm}$.

where

$$\bar{c}_\varphi = \cos \bar{\varphi}, \quad \bar{s}_\varphi = \sin \bar{\varphi}, \quad \tan 2\bar{\varphi} \equiv -\frac{ac_{13}^2 \sin 2\theta_{12}}{\delta m_{21}^2 + ac_{13}^2 \cos 2\theta_{12}}, \quad \left(-\frac{\pi}{2} < \bar{\varphi} < 0\right). \quad (\text{B.41})$$

Using \bar{V} we find

$$\bar{H}_a'' \equiv \bar{V}^\dagger \bar{H}_a' \bar{V} = \begin{bmatrix} \bar{\lambda}'_- & 0 & -a\bar{c}'_{12}c_{13}s_{13} \\ 0 & \bar{\lambda}'_+ & -a\bar{s}'_{12}c_{13}s_{13} \\ -a\bar{c}'_{12}c_{13}s_{13} & -a\bar{s}'_{12}c_{13}s_{13} & -as_{13}^2 + \delta m_{31}^2 \end{bmatrix}, \quad (\text{B.42})$$

where

$$\bar{c}'_{12} = \cos \bar{\theta}'_{12}, \quad \bar{s}'_{12} = \sin \bar{\theta}'_{12}, \quad \bar{\theta}'_{12} \equiv \theta_{12} + \bar{\varphi}, \quad (\text{B.43})$$

and

$$\bar{\lambda}'_{\pm} \equiv \frac{(\delta m_{21}^2 - ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 + ac_{13}^2)^2 - 4ac_{13}^2 s_{12}^2 \delta m_{21}^2}}{2}. \quad (\text{B.44})$$

The angle $\bar{\theta}'_{12}$ can be calculated directly without going through $\bar{\varphi}$ via

$$\tan 2\bar{\theta}'_{12} = \frac{\delta m_{21}^2 \sin 2\theta_{12}}{\delta m_{21}^2 \cos 2\theta_{12} + ac_{13}^2}, \quad \left(0 \leq \bar{\theta}'_{12} \leq \theta_{12}\right). \quad (\text{B.45})$$

The β -dependences of $\bar{\theta}'_{12}$ and $\bar{\lambda}'_{\pm}$ are shown in Fig. 15. Note that in contrast to the neutrino case, there is no level crossing. $\bar{\theta}'_{12}$ decreases monotonically toward zero as a is increased. For $a \gg \delta m_{21}^2$, \bar{s}'_{12} and \bar{c}'_{12} behave as

$$\begin{aligned} \bar{s}'_{12} &= s_{12}c_{12} \left(\frac{\delta m_{21}^2}{ac_{13}^2}\right) - s_{12}c_{12}(c_{12}^2 - s_{12}^2) \left(\frac{\delta m_{21}^2}{ac_{13}^2}\right)^2 + \dots, \\ \bar{c}'_{12} &= 1 - \frac{s_{12}^2 c_{12}^2}{2} \left(\frac{\delta m_{21}^2}{ac_{13}^2}\right)^2 + \dots, \end{aligned} \quad (\text{B.46})$$

and we see that, this time, we have $a\bar{c}'_{12} \approx a$ and $a\bar{s}'_{12} \approx \delta m_{21}^2 s_{12}c_{12}/c_{13}^2 = O(\varepsilon^2 |\delta m_{31}^2|)$. These β -dependences of \bar{s}'_{12} , \bar{c}'_{12} , $a\bar{s}'_{12}$, and $a\bar{c}'_{12}$ are shown in Fig. 16(a) and (b).

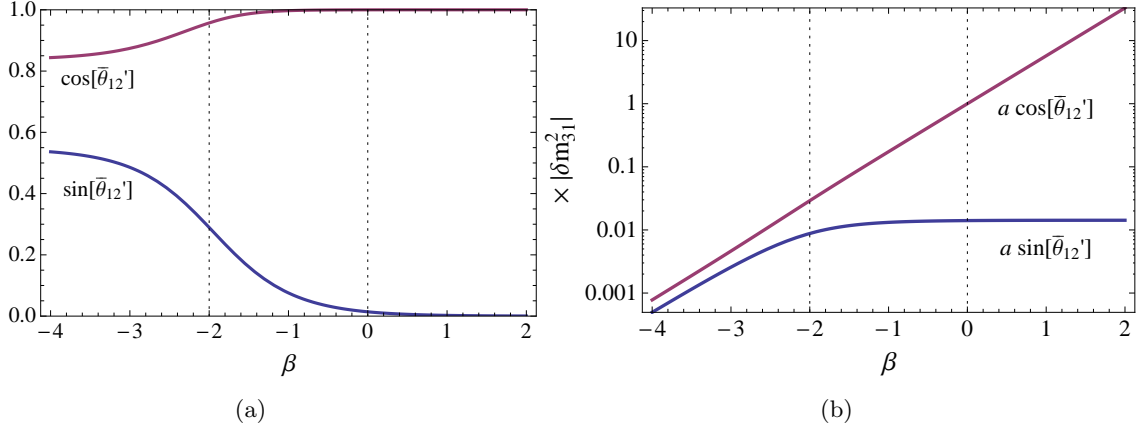


Figure 16. (a) The β -dependence of $\bar{s}'_{12} = \sin \bar{\theta}'_{12}$ and $\bar{c}'_{12} = \cos \bar{\theta}'_{12}$. (b) The β -dependence of $a \bar{s}'_{12}$ and $a \bar{c}'_{12}$. The asymptotic value of $a \bar{s}'_{12}$ is $\delta m_{21}^2 s_{12} c_{12} / c_{13}^2 \approx 0.014 |\delta m_{31}^2| = O(\varepsilon^2 |\delta m_{31}^2|)$.

In the range $a \ll \delta m_{21}^2$, $\bar{\lambda}'_{\pm}$ can be expanded as

$$\begin{aligned} \bar{\lambda}'_- &= -ac_{13}^2 c_{12}^2 \left[1 + s_{12}^2 \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right) - s_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right)^2 + \dots \right], \\ \bar{\lambda}'_+ &= \delta m_{21}^2 \left[1 - s_{12}^2 \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right) + s_{12}^2 c_{12}^2 \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right)^2 - s_{12}^2 c_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{ac_{13}^2}{\delta m_{21}^2} \right)^3 + \dots \right], \end{aligned} \quad (\text{B.47})$$

while in the range $a \gg \delta m_{21}^2$, we obtain

$$\begin{aligned} \bar{\lambda}'_- &= -ac_{13}^2 \left[1 - s_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right) + s_{12}^2 c_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^2 - s_{12}^2 c_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^3 + \dots \right], \\ \bar{\lambda}'_+ &= \delta m_{21}^2 c_{12}^2 \left[1 + s_{12}^2 \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right) - s_{12}^2 (c_{12}^2 - s_{12}^2) \left(\frac{\delta m_{21}^2}{ac_{13}^2} \right)^2 + \dots \right]. \end{aligned} \quad (\text{B.48})$$

The asymptotic values are thus $\bar{\lambda}'_+ \rightarrow \delta m_{21}^2$, $\bar{\lambda}'_- \rightarrow -ac_{13}^2 c_{12}^2$, in the limit $a \rightarrow 0$, and $\bar{\lambda}'_+ \rightarrow c_{12}^2 \delta m_{21}^2$, $\bar{\lambda}'_- \rightarrow -ac_{13}^2$, in the limit $a \rightarrow \infty$.

B.4.2 Second Rotation

After the first rotation, the effective hamiltonian was given by Eq. (B.42). When $a < \delta m_{21}^2$, both non-zero off-diagonal elements are of order $O(\varepsilon a) < O(\varepsilon^3 |\delta m_{31}^2|)$. In contrast to the neutrino case, as a increases beyond δm_{21}^2 , the angle $\bar{\theta}'_{12}$ approaches 0, and it is the 1-3 element that becomes the larger of the two. Therefore, a 1-3 rotation is needed next.

We define

$$\bar{W} = \begin{bmatrix} \bar{c}_\phi & 0 & \bar{s}_\phi \\ 0 & 1 & 0 \\ -\bar{s}_\phi & 0 & \bar{c}_\phi \end{bmatrix}, \quad (\text{B.49})$$

where

$$\bar{c}_\phi = \cos \bar{\phi}, \quad \bar{s}_\phi = \sin \bar{\phi}, \quad \tan 2\bar{\phi} \equiv -\frac{a\bar{c}'_{12} \sin 2\theta_{13}}{\delta m_{31}^2 - as_{13}^2 - \bar{\lambda}'_-}. \quad (\text{B.50})$$

The angle $\bar{\phi}$ is in the fourth quadrant when $\delta m_{31}^2 > 0$, and the first quadrant when $\delta m_{31}^2 < 0$. Using \bar{W} , we find

$$\bar{H}_a''' \equiv \bar{W}^\dagger \bar{H}_a'' \bar{W} = \begin{bmatrix} \bar{\lambda}_\mp'' & a\bar{s}'_{12}c_{13}s_{13}\bar{s}_\phi & 0 \\ a\bar{s}'_{12}c_{13}s_{13}\bar{s}_\phi & \bar{\lambda}_+'' & -a\bar{s}'_{12}c_{13}s_{13}\bar{c}_\phi \\ 0 & -a\bar{s}'_{12}c_{13}s_{13}\bar{c}_\phi & \bar{\lambda}_\pm'' \end{bmatrix}, \quad (\text{B.51})$$

where the upper(lower) sign corresponds to normal(inverted) mass hierarchy with

$$\bar{\lambda}_\pm'' \equiv \frac{[\bar{\lambda}'_- + (\delta m_{31}^2 - as_{13}^2)] \pm \sqrt{[\bar{\lambda}'_- - (\delta m_{31}^2 - as_{13}^2)]^2 + 4a^2\bar{c}'_{12}c_{13}^2s_{13}^2}}{2}. \quad (\text{B.52})$$

The β -dependence of $\bar{\lambda}_\pm''$ and $\bar{\phi}$ are shown in Fig. 4 ((c) and (d)), and Fig. 17(a), respectively, for both normal and inverted mass hierarchies. For the normal hierarchy case, $\delta m_{31}^2 > 0$, there is no level crossing, and $\bar{\lambda}_\pm''$ are well approximated by

$$\bar{\lambda}_+'' \approx \delta m_{31}^2, \quad \bar{\lambda}_-'' \approx \bar{\lambda}'_- . \quad (\text{B.53})$$

Level crossing occurs for the inverted hierarchy case, $\delta m_{31}^2 < 0$, in which we have

$$\begin{aligned} \bar{\lambda}_+'' &\approx \bar{\lambda}'_- , \\ \bar{\lambda}_-'' &\approx -\delta m_{31}^2 - as_{13}^2 , \end{aligned} \quad (\text{B.54})$$

when $a \ll \delta m_{31}^2$, and

$$\begin{aligned} \bar{\lambda}_+'' &\approx -c_{13}^2\delta m_{31}^2 + s_{13}^2s_{12}^2\delta m_{21}^2 , \\ \bar{\lambda}_-'' &\approx -a - s_{13}^2\delta m_{31}^2 + c_{13}^2s_{12}^2\delta m_{21}^2 , \end{aligned} \quad (\text{B.55})$$

when $a \gg \delta m_{31}^2$.

Here, as in the neutrino case, we approximate $\bar{\phi}$ with the angle $\bar{\phi}'$ defined via

$$\tan 2\bar{\phi}' = -\frac{a \sin 2\theta_{13}}{(\delta m_{31}^2 - s_{12}^2\delta m_{21}^2) + a \cos^2 2\theta_{13}}, \quad (\text{B.56})$$

which is obtained by using Eqs. (B.46) and (B.48) on Eq. (B.50). The difference between $\bar{\phi}'$ and $\bar{\phi}$ is shown in Fig. 17(b), and it is clear that the difference is negligible.

Now, the effective Hamiltonian after the second rotation was given by Eq. (B.51). Note that all of the non-zero off-diagonal elements include the factor $a\bar{s}'_{12}$, which is never larger than $O(\varepsilon^2|\delta m_{31}^2|)$ regardless of the value of a as discussed above. They also all include a factor of s_{13} , which is $O(\varepsilon)$ as we have seen in Eq. (B.9). Therefore, all off-diagonal elements of \bar{H}_a''' are of order $O(\varepsilon^2s_{13}|\delta m_{31}^2|) = O(\varepsilon^3|\delta m_{31}^2|)$ or smaller regardless of the size of a . We conclude that, at this point, off-diagonal elements are negligible and a third rotation is not necessary.

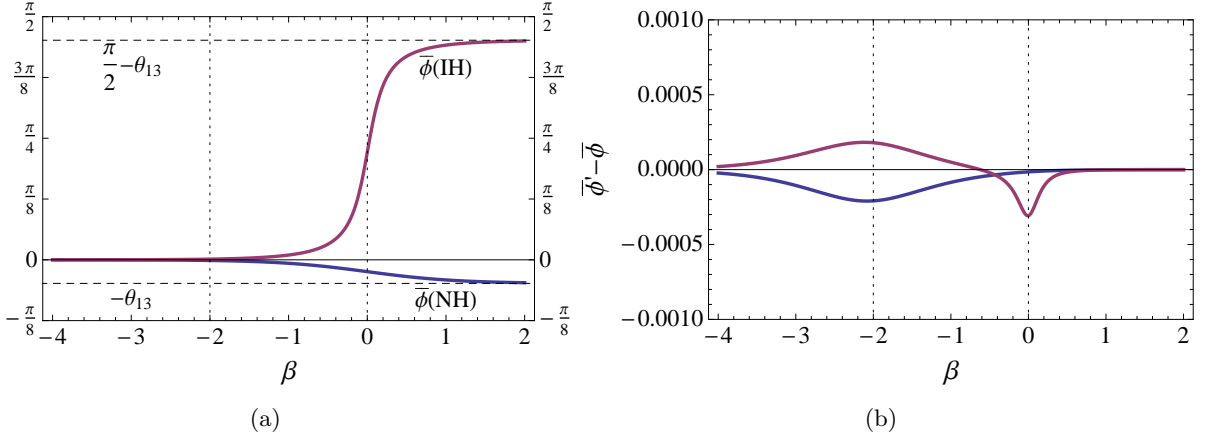


Figure 17. (a) The β -dependence of $\bar{\phi}$ for the normal and inverted hierarchies. (b) The β -dependence of the difference $\bar{\phi}' - \bar{\phi}$.

B.4.3 Absorption of $\bar{\phi}'$ into θ_{13}

From the above consideration, we conclude that the matrix which diagonalizes \bar{H}'_a , Eq. (B.39), is given approximately by $\bar{V}\bar{W}$, and that the effective anti-neutrino mixing matrix becomes

$$\tilde{U} \approx U \bar{Q} \bar{V} \bar{W} = \underbrace{R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0)}_U \underbrace{\bar{Q} R_{12}(\bar{\varphi}, 0)}_{\bar{V}} \underbrace{R_{13}(\bar{\phi}', 0)}_{\bar{W}}. \quad (\text{B.57})$$

As in the neutrino case, we find

$$\begin{aligned} \tilde{U} &\approx R_{23}(\theta_{23}, 0) R_{13}(\theta_{13}, \delta) R_{12}(\theta_{12}, 0) \bar{Q} R_{12}(\bar{\varphi}, 0) R_{13}(\bar{\phi}', 0) \\ &= R_{23}(\theta_{23}, 0) \bar{Q} R_{13}(\theta_{13}, 0) R_{12}(\theta_{12}, 0) R_{12}(\bar{\varphi}, 0) R_{13}(\bar{\phi}', 0) \\ &= R_{23}(\theta_{23}, 0) \bar{Q} R_{13}(\theta_{13}, 0) R_{12}(\theta_{12} + \bar{\varphi}, 0) R_{13}(\bar{\phi}', 0) \\ &= R_{23}(\theta_{23}, 0) \bar{Q} R_{13}(\theta_{13}, 0) R_{12}(\bar{\theta}'_{12}, 0) R_{13}(\bar{\phi}', 0). \end{aligned} \quad (\text{B.58})$$

Here, we argue that

$$R_{12}(\bar{\theta}'_{12}, 0) R_{13}(\bar{\phi}', 0) \approx R_{13}(\bar{\phi}', 0) R_{12}(\bar{\theta}'_{12}, 0), \quad (\text{B.59})$$

that is, the 1-3 rotation passes through $R_{12}(\bar{\theta}'_{12}, 0)$. This is due to the fact that $\bar{\phi}'$ only becomes non-negligible when $a \gg \delta m_{12}^2$ where $\bar{s}'_{12} \approx 0$ and $\bar{c}'_{12} \approx 1$, which means

$$R_{12}(\bar{\theta}'_{12}, 0) \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\text{B.60})$$

thus any matrix will commute with $R_{12}(\bar{\theta}'_{12}, 0)$. In the range $a \lesssim \delta m_{21}^2$, the angle ϕ' is very small and both $R_{23}(\phi', 0)$ and $R_{13}(\phi', 0)$ are approximately unit matrices and Eq. (B.59) is trivially satisfied. The accuracy of this approximation is discussed in appendix C. Therefore,

$$\tilde{U} \approx R_{23}(\theta_{23}, 0) \bar{Q} R_{13}(\theta_{13}, 0) R_{12}(\bar{\theta}'_{12}, 0) R_{13}(\bar{\phi}', 0)$$

$$\begin{aligned}
&\approx R_{23}(\theta_{23}, 0) \mathcal{Q} R_{13}(\theta_{13}, 0) R_{13}(\bar{\phi}', 0) R_{12}(\bar{\theta}'_{12}, 0) \\
&= R_{23}(\theta_{23}, 0) \mathcal{Q} R_{13}(\theta_{13} + \bar{\phi}', 0) R_{12}(\bar{\theta}'_{12}, 0) \\
&= R_{23}(\theta_{23}, 0) \mathcal{Q} R_{13}(\bar{\theta}'_{13}, 0) R_{12}(\bar{\theta}'_{12}, 0) \\
&= R_{23}(\theta_{23}, 0) R_{13}(\bar{\theta}'_{13}, \delta) R_{12}(\bar{\theta}'_{12}, 0) \mathcal{Q},
\end{aligned} \tag{B.61}$$

where we have defined

$$\bar{\theta}'_{13} \equiv \theta_{13} + \bar{\phi}'. \tag{B.62}$$

This angle can be calculated directly without calculation $\bar{\phi}'$ via

$$\tan 2\bar{\theta}'_{13} = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} + a}. \tag{B.63}$$

The phase matrix \mathcal{Q} appearing rightmost in the above matrix product can be absorbed into the redefinition of the major phases and can be dropped. Thus, we arrive at our final approximation in which the vacuum mixing angles are replaced by their effective values in matter

$$\begin{aligned}
\theta_{12} &\rightarrow \bar{\theta}'_{12} = \theta_{12} + \bar{\varphi}, \\
\theta_{13} &\rightarrow \bar{\theta}'_{13} = \theta_{13} + \bar{\phi}', \\
\theta_{23} &\rightarrow \theta_{23}, \\
\delta &\rightarrow \delta,
\end{aligned} \tag{B.64}$$

and the eigenvalues of the effective Hamiltonian are given by

$$\begin{aligned}
\bar{\lambda}_1 &\approx \bar{\lambda}_{\mp}'' , \\
\bar{\lambda}_2 &\approx \bar{\lambda}_{+}' , \\
\bar{\lambda}_3 &\approx \bar{\lambda}_{\pm}'' .
\end{aligned} \tag{B.65}$$

Note that of the mixing angles, only θ_{12} and θ_{13} are shifted. θ_{23} and δ stay at their vacuum values.

C Commutation of R_{13} and R_{23} through R_{12}

In the derivation of our approximation formulae above, Eqs. (B.31) and (B.59) played crucial roles in allowing the second rotation angle to be absorbed into θ_{13} . In this appendix, we evaluate the validity of these approximations.

C.1 Neutrino Case

The difference between the two sides of Eq. (B.31) is given by

$$\begin{aligned}
\delta R &\equiv R_{12}(\theta'_{12}, 0) R_{23}(\phi', 0) - R_{13}(\phi', 0) R_{12}(\theta'_{12}, 0) \\
&= \begin{bmatrix} c'_{12}(1 - c'_{\phi}) & 0 & -(1 - s'_{12})s'_{\phi} \\ 0 & -c'_{12}(1 - c'_{\phi}) & c'_{12}s'_{\phi} \\ c'_{12}s'_{\phi} & -(1 - s'_{12})s'_{\phi} & 0 \end{bmatrix}.
\end{aligned} \tag{C.1}$$

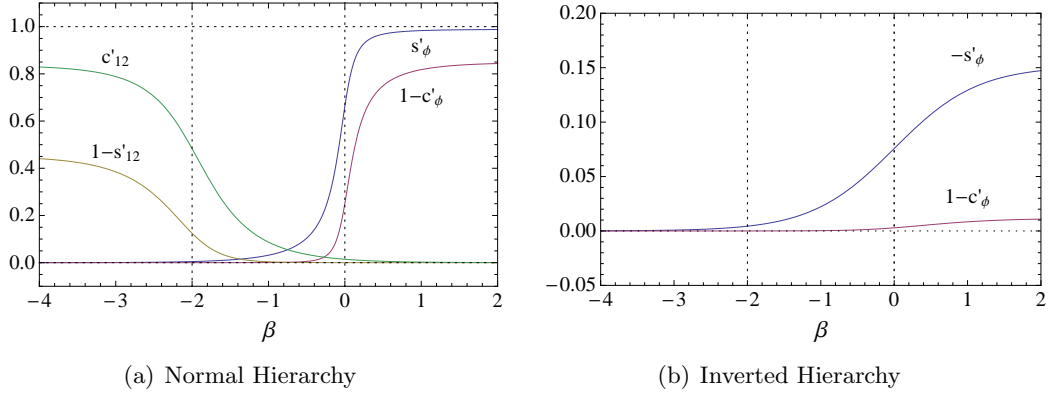


Figure 18. β -dependence of c'_{12} , $1 - s'_{12}$, s'_ϕ and $1 - c'_\phi$ for (a) normal and (b) inverted hierarchies. The behaviors of c'_{12} and $1 - s'_{12}$ are common to both.

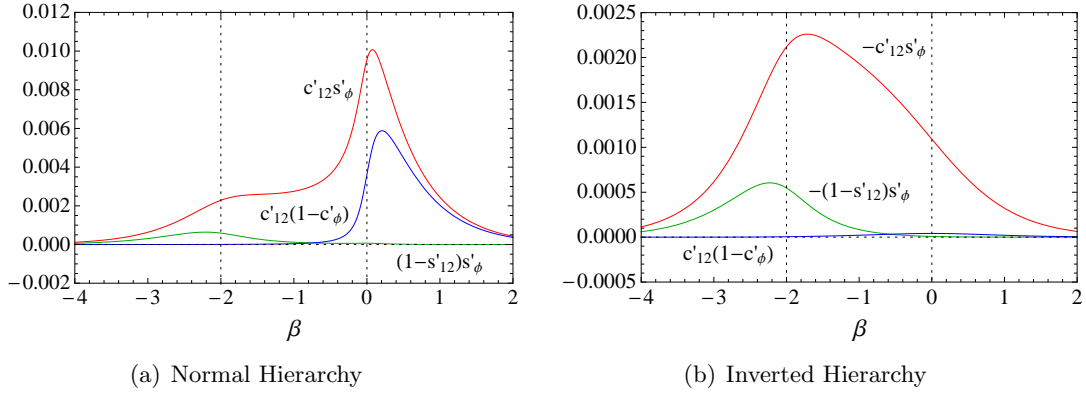


Figure 19. β -dependence of the non-zero elements of δR for (a) normal and (b) inverted hierarchies.

It is clear that δR will vanish in the two limits $a \rightarrow 0$ where $s'_{12} \rightarrow s_{12}$, $c'_{12} \rightarrow c_{12}$, $s'_\phi \rightarrow 0$, and $c'_\phi \rightarrow 1$, and $a \rightarrow \infty$ where $s'_{12} \rightarrow 1$, $c'_{12} \rightarrow 0$, $s'_\phi \rightarrow c_{13}(-s_{13})$, and $c'_\phi \rightarrow s_{13}(c_{13})$ for normal(inverted) hierarchy. The question is whether δR will stay negligible in between as s'_{12} runs from s_{12} to 1, c'_{12} from c_{12} to 0, s'_ϕ from 0 to c_{13} (normal) or $-s_{13}$ (inverted), and c'_ϕ from 1 to s_{13} (normal) or c_{13} (inverted) as shown in Fig. 18. The dependence of the non-zero elements of δR on $\beta = -\log_\epsilon(a/|\delta m_{31}^2|)$ is shown in Fig. 19. The bumps at $a \sim \delta m_{21}^2$ for both hierarchies, and that at $a \sim \delta m_{31}^2$ for the normal hierarchy, happen due to the θ'_{12} factor competing with the ϕ' factor as one of them goes through a resonance while the other damps to zero. The heights of the bumps depend on the narrowness of the resonances.

For the case shown in Fig. 19, which was generated with the numbers in Table 1 as input, all elements of δR are $O(\epsilon^3)$ or smaller for the entire range of a , with the maximum value of $\sim 0.01 \approx 2\epsilon^3$ occurring in $c'_{12}s'_\phi$ near $a \sim \delta m_{31}^2$ in the normal hierarchy case. Since the size of the third rotation angle we neglected in the Jacobi procedure was $O(\epsilon^2 s_{13})$, Eq. (B.31) is valid to the same order provided $s_{13} = O(\epsilon)$.

For smaller values of s_{13} , the resonance at $a \sim \delta m_{31}^2$ would have been narrower, and the

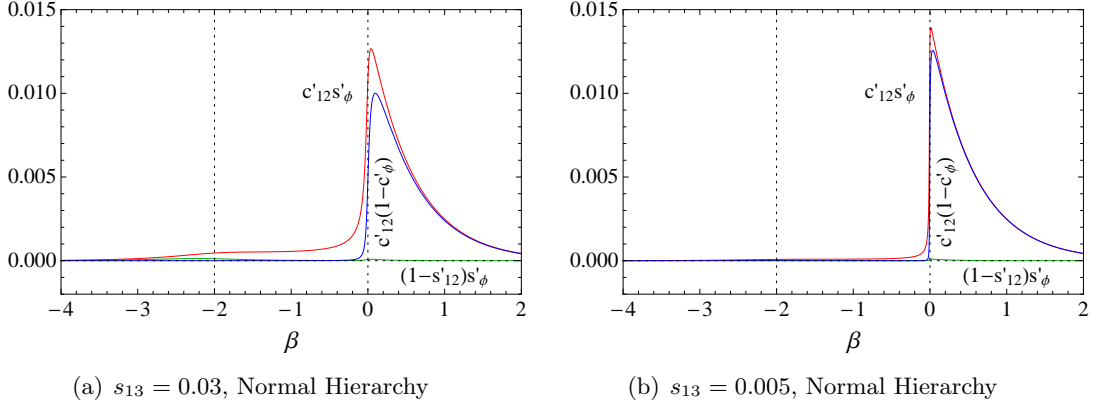


Figure 20. β -dependence of the non-zero elements of δR for different values of s_{13} with normal hierarchy. (a) $s_{13} = 0.03 = O(\varepsilon^2)$, (b) $s_{13} = 0.005 = O(\varepsilon^3)$.

peaks in $c'_{12}s'_\phi$ and $c'_{12}(1-c'_\phi)$ higher. This is illustrated in Fig. 20. In the limit $s_{13} \rightarrow +0$, s'_ϕ and $1-c'_\phi$ will become step functions at $\beta \sim 0$, and the maximum height of the peak will be

$$c'_{12}(a \sim \delta m_{31}^2) \approx s_{12}c_{12}\varepsilon^2 = 0.46\varepsilon^2 = 0.014, \quad (\text{C.2})$$

as can be discerned from Eq. (B.17). This is the same as the asymptotic value of $ac'_{12}/\delta m_{31}^2$ discussed earlier. While this value may not seem particularly large, only a factor of $3/2$ larger than the peak in Fig. 19(a), it is parametrically $O(\varepsilon^2)$. On the other hand, the third rotation angle neglected in the Jacobi procedure was $O(\varepsilon^2 s_{13})$. Thus, using Eq. (B.31) would lead to dropping terms that are larger than the ones we keep when $s_{13} = O(\varepsilon^2)$ or smaller. Also, the sudden change in the accuracy of Eq. (B.31) across $a \sim \delta m_{31}^2$, as can be seen in Fig. 20, will lead to kinks in the resulting oscillation probabilities.

C.2 Anti-neutrino Case

The difference between the two sides of Eq. (B.59) is given by

$$\begin{aligned} \overline{\delta R} &\equiv R_{12}(\bar{\theta}'_{12}, 0)R_{13}(\bar{\phi}', 0) - R_{13}(\bar{\phi}', 0)R_{12}(\bar{\theta}'_{12}, 0) \\ &= \begin{bmatrix} 0 & \bar{s}'_{12}(1-\bar{c}'_\phi) & -(1-\bar{c}'_{12})\bar{s}'_\phi \\ \bar{s}'_{12}(1-\bar{c}'_\phi) & 0 & -\bar{s}'_{12}\bar{s}'_\phi \\ (1-\bar{c}'_{12})\bar{s}'_\phi & \bar{s}'_{12}\bar{s}'_\phi & 0 \end{bmatrix}. \end{aligned} \quad (\text{C.3})$$

It is clear that $\overline{\delta R}$ will vanish in the two limits $a \rightarrow 0$ where $\bar{s}'_{12} \rightarrow s_{12}$, $\bar{c}'_{12} \rightarrow c_{12}$, $\bar{s}'_\phi \rightarrow 0$, and $\bar{c}'_\phi \rightarrow 1$, and $a \rightarrow \infty$ where $\bar{s}'_{12} \rightarrow 0$, $\bar{c}'_{12} \rightarrow 1$, $\bar{s}'_\phi \rightarrow -s_{13}(c_{13})$, and $\bar{c}'_\phi \rightarrow c_{13}(s_{13})$ for normal(inverted) hierarchy. The question is whether δR will stay negligible in between as \bar{s}'_{12} runs from s_{12} to 0, \bar{c}'_{12} from c_{12} to 1, \bar{s}'_ϕ from 0 to $-s_{13}$ (normal) or c_{13} (inverted), and \bar{c}'_ϕ from 1 to c_{13} (normal) or s_{13} (inverted) as shown in Fig. 21.

The dependence of the non-zero elements of $\overline{\delta R}$ on $\beta = -\log_\varepsilon(a/|\delta m_{31}^2|)$ is shown in Fig. 22, which was generated with the numbers in Table 1 as input. We can see that all

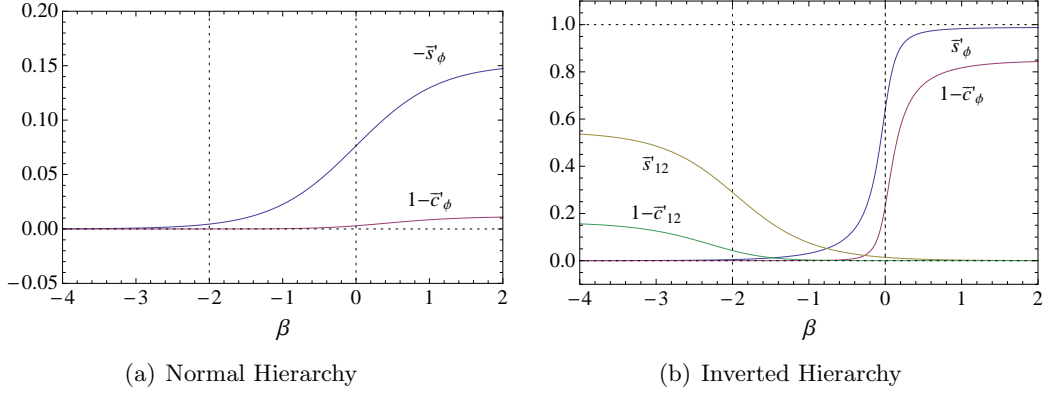


Figure 21. β -dependence of \bar{s}'_{12} , $1-\bar{c}'_{12}$, s'_ϕ and $1-c'_\phi$ for (a) normal and (b) inverted hierarchies. The behaviors of \bar{s}'_{12} and $1-\bar{c}'_{12}$ are common to both.

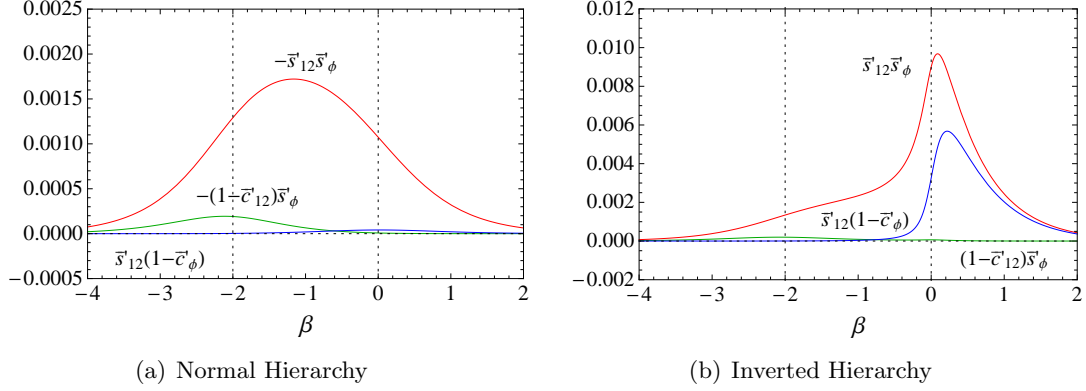


Figure 22. β -dependence of the non-zero elements of $\overline{\delta R}$ for (a) normal and (b) inverted hierarchies.

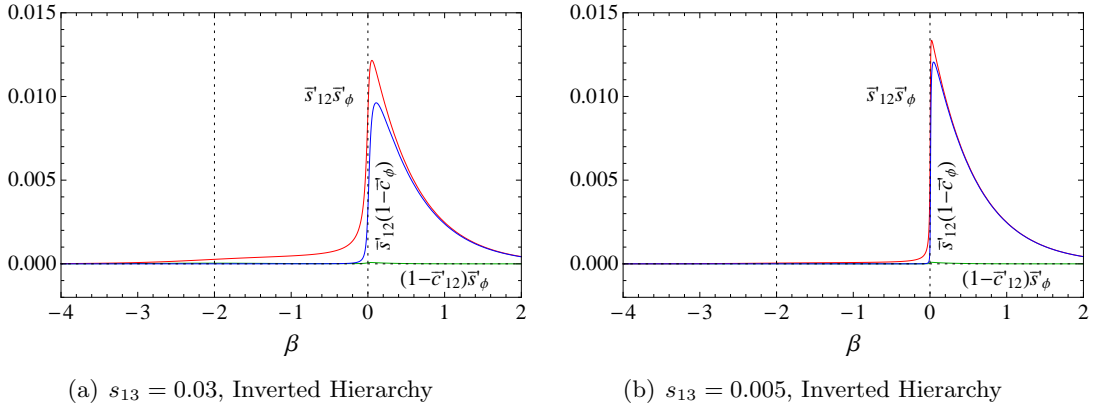


Figure 23. β -dependence of the non-zero elements of $\overline{\delta R}$ for different values of s_{13} with inverted hierarchy. (a) $s_{13} = 0.03 = O(\varepsilon^2)$, (b) $s_{13} = 0.005 = O(\varepsilon^3)$.

elements of $\overline{\delta R}$ are $O(\varepsilon^3)$ or smaller for the entire range of a , with the maximum value of

$\sim 0.01 \approx 2\varepsilon^3$ occurring in $\bar{s}'_{12}\bar{s}'_\phi$ near $a \sim \delta m_{31}^2$ in the inverted hierarchy case. Since the size of the third rotation angle we neglected in the Jacobi procedure was $O(\varepsilon^2 s_{13})$, Eq. (B.59) is valid to the same order provided $s_{13} = O(\varepsilon)$.

For smaller values of s_{13} , the resonance at $a \sim \delta m_{31}^2$ would have been narrower, and the peaks in $\bar{s}'_{12}\bar{s}'_\phi$ and $\bar{s}'_{12}(1 - \bar{c}'_\phi)$ higher. This is illustrated in Fig. 23. In the limit $s_{13} \rightarrow +0$, \bar{s}'_ϕ and $1 - \bar{c}'_\phi$ will become step functions at $\beta \sim 0$, and the maximum height of the peak will be the same as Eq. (C.2), and the asymptotic value of $a\bar{s}'_{12}/|\delta m_{31}^2|$, as can be discerned from Eq. (B.46). This is parametrically $O(\varepsilon^2)$, while the third rotation angle neglected in the Jacobi procedure was $O(\varepsilon^2 s_{13})$. Thus, using Eq. (B.59) would lead to dropping terms that are larger than the ones we keep when $s_{13} = O(\varepsilon^2)$ or smaller. Also, the sudden change in the accuracy of Eq. (B.59) across $a \sim \delta m_{31}^2$, as can be seen in Fig. 23, will lead to kinks in the resulting oscillation probabilities.

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